Chapter 4

Surfaces in $\mathbf{R}^3$

4.1 Manifolds

4.1 Definition A C^∞ coordinate chart is a C^∞ map x from an open subset of $\mathbf{R}^2$ into $\mathbf{R}^3$.

$$\begin{aligned} \mathbf{x} : U \in \mathbf{R}^2 &\longrightarrow \mathbf{R}^3 \\ (u, v) &\stackrel{\mathbf{x}}{\longmapsto} (x(u,v), y(u,v), z(u,v)) \end{aligned} \tag{4.1}$$

We will always assume that the Jacobian of the map has maximal rank. In local coordinates, a coordinate chart is represented by three equations in two variables

$$x^i = f^i(u^\alpha), \text{ where } i = 1, 2, 3, \ \alpha = 1, 2. \tag{4.2}$$

The local coordinate representation allows us to use the tensor index formalism introduced in earlier chapters. The assumption the Jacobian $J = (\partial x^i / \partial u^\alpha)$ be of maximal rank, allows one to evoke the implicit function theorem. Thus, in principle, one can locally solve for one of the coordinates, say x^3 in terms of the other two

$$x^3 = f(x^1, x^2). \tag{4.3}$$

The locus of points in $\mathbf{R}^3$ satisfying the equations $x^i = f^i(u^\alpha)$, can also be locally represented by an expression of the form

$$F(x^1, x^2, x^3) = 0 \tag{4.4}$$

4.2 Definition Let $\mathbf{x}(u^1, u^2) : U \longrightarrow \mathbf{R}^3$ and $\mathbf{y}(v^1, v^2) : V \longrightarrow \mathbf{R}^3$ be two coordinate charts with a non empty intersection $(\mathbf{x}(U) \cap \mathbf{y}(V)) \neq \emptyset$. The two charts are said to be C^∞ equivalent if the map $\phi = \mathbf{y}^{-1}\mathbf{x}$ and its inverse ϕ^{-1} (see fig 4.1)are infinitely differentiable.

In more lucid terms, the definition just states that two equivalent charts $\mathbf{x}(u^\alpha)$ and $\mathbf{y}(v^\beta)$ represent different reparametrizations for the same set of points in $\mathbf{R}^3$.

Figure 4.1: Chart Equivalence

4.3 Definition A differentiably smooth surface in $\mathbf{R}^3$ is a set of points $\mathcal{M}$ in $\mathbf{R}^3$ such that

1. If $\mathbf{p} \in \mathcal{M}$ then $\mathbf{p}$ belongs to some C^∞ chart.
2. If $\mathbf{p} \in \mathcal{M}$ belongs to two different charts $\mathbf{x}$ and $\mathbf{y}$, then the two charts are C^∞ equivalent.

Intuitively, we may think of a surface as consisting locally of number of patches "sewn" to each other so as to form a quilt from a global perspective.

The first condition in the definition states that each local patch looks like a piece of $\mathbf{R}^2$, whereas the second differentiabilty condition indicates that the patches are joined together smoothly. Another way to state this idea is to say that a surface a space that is locally Euclidean and it has a differentiable structure so that the notion of differentiation makes sense. If the Euclidean space is of dimension n, the "surface" is called an n-dimensional **manifold**

4.4 Example Consider the local coordinate chart

$$\mathbf{x}(u,v) = (\sin u \cos v, \sin u \sin v, \cos v).$$

The vector equation is equivalent to three scalar functions in two variables

$$\begin{aligned} x &= \sin u \cos v, \\ y &= \sin u \sin v, \\ z &= \cos u. \end{aligned} \tag{4.5}$$

Clearly, the surface represented by this chart is part of the sphere $x^2+y^2+z^2=1$. The chart can not possibly represent the whole sphere because, although a sphere is locally Euclidean, (the earth is locally flat) there is certainly a topological difference between a sphere and a plane. Indeed, if one analyzes the coordinate chart carefully you will note that at the North pole ($u=0$, $z=1$, the coordinates become singular. This happens because $u=0$ implies that $x=y=0$ regardless of the value of v, so that the North pole has an infinite number of labels. The fact that it is required to have two parameters to describe a patch on a surface in $\mathbf{R}^3$ is a manifestation of the 2-dimensional nature of of the surfaces. If one holds one of the parameters constant while varying the other, then the resulting 1-parameter equations describe a curve on the surface. Thus, for example, letting $u = constant$ in equation (4.5) we get the equation of a meridian great circle.

4.5 Notation Given a parametrization of a surface in a local chart $\mathbf{x}(u,v) = \mathbf{x}(u^1,u^2) = \mathbf{x}(u^\alpha)$, we will denote the partial derivatives by any of the following notations:

$$\begin{aligned} \mathbf{x}_u = \mathbf{x}_1 &= \frac{\partial \mathbf{x}}{\partial u}, & \mathbf{x}_{uu} = \mathbf{x}_{11} &= \frac{\partial^2 \mathbf{x}}{\partial u^2} \\ \mathbf{x}_v = \mathbf{x}_v &= \frac{\partial \mathbf{x}}{\partial v}, & \mathbf{x}_{vv} = \mathbf{x}_{22} &= \frac{\partial^2 \mathbf{x}}{\partial v^2} \\ \mathbf{x}_\alpha &= \frac{\partial \mathbf{x}}{\partial u^\alpha} & \mathbf{x}_{\alpha\beta} &= \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial v^\beta} \end{aligned}$$

4.2 The First Fundamental Form

Let $x^i(u^\alpha)$ be a local parametrization of a surface. Then, the Euclidean inner product in $\mathbf{R}^3$ induces an inner product in the space of tangent vectors at each point in the surface. This metric on the surface is obtained as follows:

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha$$

$$\begin{aligned} ds^2 &= \delta_{ij}dx^i dx^j \\ &= \delta_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}du^\alpha du^\beta. \end{aligned}$$

Thus,

$$ds^2 = g_{\alpha\beta}du^\alpha du^\beta, \tag{4.6}$$

where

$$g_{\alpha\beta} = \delta_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}. \tag{4.7}$$

We conclude that the surface, by virtue of being embedded in $\mathbf{R}^3$, inherits a natural metric (4.6) which we will call the **induced metric**. A pair $\{\mathcal{M}, g\}$, where $\mathcal{M}$ is a manifold and $g = g_{\alpha\beta}du^\alpha \otimes du^\beta$ is a metric is called a **Riemannian manifold** if considered as an entity in itself, and Riemannian submanifold of $\mathbf{R}^n$ if viewed as an object embedded in Euclidean space. An equivalent version of the metric (4.6) can be obtained by using a more traditional calculus notation

$$\begin{aligned} d\mathbf{x} &= \mathbf{x}_u du + \mathbf{x}_v dv \\ ds^2 &= d\mathbf{x}\cdot d\mathbf{x} \\ &= (\mathbf{x}_u du + \mathbf{x}_v dv)\cdot(\mathbf{x}_u du + \mathbf{x}_v dv) \\ &= (\mathbf{x}_u\cdot\mathbf{x}_u)du^2 + 2(\mathbf{x}_u\cdot\mathbf{x}_v)dudv(\mathbf{x}_v\cdot\mathbf{x}_v)dv^2. \end{aligned}$$

We can rewrite the last result as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2, \tag{4.8}$$

where

$$\begin{aligned} E &= g_{11} = \mathbf{x}_u\cdot\mathbf{x}_u \\ F &= g_{12} = \mathbf{x}_u\cdot\mathbf{x}_v \\ &= g_{21} = \mathbf{x}_v\cdot\mathbf{x}_u \\ G &= g_{22} = \mathbf{x}_v\cdot\mathbf{x}_v. \end{aligned}$$

That is

$$g_{\alpha\beta} = \mathbf{x}_\alpha\cdot\mathbf{x}_\beta =< \mathbf{x}_\alpha, \mathbf{x}_\beta > .$$

4.6 bdf The element of arclength

$$ds^2 = g_{\alpha\beta}du^\alpha \otimes du^\beta \tag{4.9}$$

is also called the **first fundamental form**. We must caution the reader that this quantity is not a form in the sense of differential geometry since ds^2 involves the symmetric tensor product rather than the wedge product.

The first fundamental form plays such a crucial role in the theory of surfaces, that will find it convenient to introduce yet a third more modern version. Following the same development as in the theory of curves, consider a surface $\mathcal{M}$ defined locally by a function $(u^1, u^2) \longmapsto \alpha(u^1, u^2)$. We say that a quantity X_p is a tangent vector at a point $\mathbf{p} \in \mathcal{M}$, if X_p is a linear derivation on the space of C^∞ real-valued functions $\{f | f : \mathcal{M} \longrightarrow \mathbf{R}\}$ on the surface. The set of all tangent vectors at a point $\mathbf{p} \in \mathcal{M}$ is called the tangent space $T_p\mathcal{M}$. As before, a vector field X on the surface is a smooth choice of a tangent vector at each point on the surface and the union of all tangent spaces is called the tangent bundle $T\mathcal{M}$.

The coordinate chart map

$$\alpha : \mathbf{R}^2 \longrightarrow \mathcal{M} \in \mathbf{R}^3$$

induces a push-forward map

$$\alpha_* : T\mathbf{R}^2 \longrightarrow T\mathcal{M}$$

defined by

$$\alpha_*(V)(f) \mid_{\alpha(u^\alpha)} = V(\alpha \circ f) \mid_{u^\alpha}$$

Just as in the case of curves, when we revert back to classical notation to describe a surface as $x^i(u^\alpha)$, what we really mean is $(x^i \circ \alpha)(u^\alpha)$, where x^1 are the coordinate functions in $\mathbf{R}^3$. Particular examples of tangent vectors on $\mathcal{M}$ are given by the push-forward of the standard basis of $T\mathbf{R}^2$. These tangent vectors which earlier we called $\mathbf{x}_\alpha$ are defined by

$$\alpha_*(\frac{\partial}{\partial u^\alpha})(f) \mid_{\alpha(u^\alpha)} = \frac{\partial}{\partial u^\alpha}(\alpha \circ f) \mid_{u^\alpha}$$

In this formalism, the first fundamental form I is just the symmetric bilinear tensor defined by induced metric

$$I(X, Y) = g(X, Y) = < X, Y >, \tag{4.10}$$

where X and Y are any pair of vector fields in $T\mathcal{M}$.

Orthogonal Parametric Curves

Let V and W be vectors tangent to a surface $\mathcal{M}$ defined locally by a chart $\mathbf{x}(u^\alpha)$. Since the vectors $\mathbf{x_{alpha}}$ span the tangent space of $\mathcal{M}$ at each point, the vectors V and W can be written as linear combinations

$$\begin{aligned} V &= V^\alpha \mathbf{x}_\alpha \\ W &= W^\alpha \mathbf{x}_\alpha. \end{aligned}$$

The functions V^α and W^α are called the curvilinear coordinates of the vectors. We can calculate the length and the inner product of the vectors using the induced Riemannian metric,

$$\begin{aligned} \|V\|^2 &= < V, V > = < V^\alpha \mathbf{x}_\alpha, V^\beta \mathbf{x}_\beta > = V^\alpha V^\beta < \mathbf{x}_\alpha, \mathbf{x}_\beta > \\ \|V\|^2 &= g_{\alpha\beta} V^\alpha V^\beta \\ \|W\|^2 &= g_{\alpha\beta} W^\alpha W^\beta, \end{aligned}$$

and

$$\begin{aligned} < V, W > &= < V^\alpha \mathbf{x}_\alpha, W^\beta \mathbf{x}_\beta > = V^\alpha W^\beta < \mathbf{x}_\alpha, \mathbf{x}_\beta > \\ &= g_{\alpha\beta} V^\alpha W^\beta. \end{aligned}$$

The angle θ subtended by the the vectors v and W is the given by the equation

$$\begin{aligned} \cos\theta &= \frac{< V, W >}{\|V\| \cdot \|W\|} \\ &= \frac{I(V, W)}{\sqrt{I(V, V)}\sqrt{I(W, W)}} \\ &= \frac{g_{\alpha\beta} V^\alpha W^\beta}{g_{\alpha\beta} V^\alpha V^\beta \cdot g_{\alpha\beta} W^\alpha W^\beta}. \end{aligned} \tag{4.11}$$

Let $u^\alpha = \phi^\alpha(t)$ and $u^\alpha = \psi^\alpha(t)$ be two curves on the surface. Then the total differentials

$$du^\alpha = \frac{d\phi^\alpha}{dt} dt, \quad \text{and} \quad \delta u^\alpha = \frac{d\psi^\alpha}{dt} \delta t$$

represent infinitesimal tangent vectors (1.12) to the curves. Thus, the angle between two infinitesimal vectors tangent to two intersecting curves on the surface satisfies

$$\cos\theta = \frac{g_{\alpha\beta}du^\alpha \delta u^\beta}{\sqrt{g_{\alpha\beta}du^\alpha du^\beta}\sqrt{g_{\alpha\beta}\delta u^\alpha \delta u^\beta}} \tag{4.12}$$

In particular, if the two curves happen to be the parametric curves, $u^1 = constant$ and $u^2 = constant$ then along one curve we have $du^1 = 0$, du^2 arbitrary, and along the second we have δu^1 arbitrary and $\delta u^2 = 0$. In this case, the cosine of the angle subtended by the infinitesimal tangent vectors reduces to

$$\cos\theta = \frac{g_{12}\delta u^1 du^2}{\sqrt{g_{11}(\delta u^1)^2}\sqrt{g_{22}(du^2)^2}} = \frac{g_{12}}{g_{11}g_{22}} = \frac{F}{\sqrt{EG}}. \tag{4.13}$$

As a result, we have the following

4.7 Proposition The parametric lines are orthogonal if $F = 0$.

4.8 Examples

a) Sphere

$$\begin{aligned}
\mathbf{x} &= (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta) \\
\mathbf{x}_\theta &= (a\cos\theta\cos\phi, a\cos\theta\sin\phi, -a\sin\theta) \\
\mathbf{x}_\phi &= (-a\sin\theta\sin\phi, a\sin\theta\cos\phi,) \\
E &= \mathbf{x}_\theta \cdot \mathbf{x}_\theta = a^2 \\
F &= \mathbf{x}_\theta \cdot \mathbf{x}_\phi = 0 \\
G &= \mathbf{x}_\phi \cdot \mathbf{x}_\phi = a^2\sin^2\theta \\
ds^2 &= a^2 d\theta^2 + a^2\sin^2\theta d\phi^2
\end{aligned}$$

b) Surface of Revolution

$$\begin{aligned}
\mathbf{x} &= (r\cos\theta, r\sin\theta, f(r)) \\
\mathbf{x}_r &= (\cos\theta, \sin\theta, f'(r)) \\
\mathbf{x}_\theta &= (-r\sin\theta, r\cos\theta, 0) \\
E &= \mathbf{x}_r \cdot \mathbf{x}_r = 1 + f'^2(r) \\
F &= \mathbf{x}_r \cdot \mathbf{x}_\theta = 0 \\
G &= \mathbf{x}_\theta \cdot \mathbf{x}_\theta = r^2 \\
ds^2 &= [1 + f'^2(r)]dr^2 + r^2 d\theta^2
\end{aligned}$$

c) Pseudosphere

$$\begin{aligned}
\mathbf{x} &= (a\sin u\cos v, a\sin u\sin v, a(\cos u \ln tan\frac{u}{2})) \\
E &= a^2\cot^2 u \\
F &= = 0 \\
G &= a^2\sin^2 u \\
ds^2 &= a^2\cot^2 u du^2 + a^2\sin^2 u dv^2
\end{aligned}$$

d) Torus

$$\begin{aligned}
\mathbf{x} &= ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u) \\
E &= a^2 \\
F &= 0 \\
G &= (b + a\cos u)^2 \\
ds^2 &= a^2 du^2 + (b + a\cos u)^2 dv^2
\end{aligned}$$

e) Helicoid

$$\begin{aligned}
\mathbf{x} &= (u\cos v, u\sin v, av) \\
E &= 1 \\
F &= 0 \\
G &= u^2 + a^2 \\
ds^2 &= du^2 + (u^2 + a^2)dv^2
\end{aligned}$$

f) Catenoid

$$\begin{aligned}
\mathbf{x} &= (u\cos v, u\sin v, c\cosh^{-1}\frac{u}{c}) \\
E &= \frac{u^2}{u^2 - c^2} \\
F &= 0 \\
G &= u^2 \\
ds^2 &= \frac{u^2}{u^2 - c^2}du^2 + u^2 dv^2
\end{aligned}$$

4.3 The Second Fundamental Form

Let $\mathbf{x} = \mathbf{x}(u^\alpha)$ be a coordinate patch on a surface $\mathcal{M}$. Since $\mathbf{x}_u$ and $\mathbf{x}v$ are tangential to the surface, we can construct a unit normal $\mathbf{n}$ to the surface by taking

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} \tag{4.14}$$

Now, consider a curve on the surface given by $u^\alpha = u^\alpha(s)$. Without loss of generality, we assume that the curve is parametrized by arclength s so that the curve has unit speed. Using the chain rule, we se that the unit tangent vector T to the curve is given by

$$T = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{du^\alpha}\frac{du^\alpha}{ds} = \mathbf{x}_\alpha \frac{du^\alpha}{ds} \tag{4.15}$$

Since the curve lives on the surface and the the vector T is tangent to the curve, it is clear that T is also tangent to the surface. On the other hand, the vector $T' = dT/ds$ does not in general have this property, so what we will do is to decompose T' into its normal and tangential components (see fig (4.2))

$$\begin{aligned}
T' &= K_n + K_g \\
&= \kappa_n \mathbf{n} + K_g,
\end{aligned} \tag{4.16}$$

where $\kappa_n = \|K_n\| = < T', \mathbf{n} >$

The scalar quantity κ_n is called the **normal curvature** of the curve and K_g is called the **geodesic curvature** vector. The normal curvature measures the the curvature of $\mathbf{x}(u^\alpha(s))$ resulting

Figure 4.2: Normal Curvature

by the constraint of the curve to lie on a surface. The geodesic curvature vector , measures the "sideward" component of the curvature in the tangent plane to the surface. Thus, if one draws a straight line on a flat piece of paper and then smoothly bend the paper into a surface, then the straight line would now acquire some curvature. Since the line was originally straight, there is no sideward component of curvature so $K_g = 0$ in this case. This means that the entire contribution to the curvature comes from the normal component, reflecting the fact that the only reason there is curvature here is due to the bend in the surface itself.

Similarly, if one specifies a point $\mathbf{p} \in \mathcal{M}$ and a direction vector $X_p \in T_p\mathcal{M}$, one can geometrically envision the normal curvature by considering the equivalence class of all unit speed curves in $\mathcal{M}$ which contain the point $\mathbf{p}$ and whose tangent vectors line up with the direction of X. Of course, there are infinitely many such curves, but at an infinitesimal level, all these curves can be obtained by intersecting the surface with a "vertical" plane containing the vector X and the normal to $\mathcal{M}$. All curves in this equivalence class have the same normal curvature and their geodesic curvatures vanish. In this sense, the normal curvature is more of a property pertaining to a direction on the surface at a point, whereas, the geodesic curvature really depends on the curve itself. It might be impossible for a hiker walking on the ondulating hills of the Ozarks to find a straight line trail, since the rolling hills of the terrain extend in all directions. However, it might be possible to walk on a path with zero geodesic curvature as long as the hiker can maintain the same compass direction.

To find an explicit formula for the normal curvature we first differentiate equation (4.15)

$$\begin{aligned}
T' &= \frac{dT}{ds} \\
&= \frac{d}{ds}(\mathbf{x}_\alpha \frac{du^\alpha}{ds}) \\
&= \frac{d}{ds}(\mathbf{x}_\alpha)\frac{du^\alpha}{ds} + \mathbf{x}_\alpha \frac{d^2u^\alpha}{ds^2} \\
&= (\frac{d\mathbf{x}_\alpha}{du^\beta}\frac{du^\beta}{ds})\frac{du^\alpha}{ds} + \mathbf{x}_\alpha \frac{d^2u^\alpha}{ds^2} \\
&= \mathbf{x}_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} + \mathbf{x}_\alpha \frac{d^2u^\alpha}{ds^2}.
\end{aligned}$$

Taking the inner product of the last equation with the normal and noticing that $< \mathbf{x}_\alpha, \mathbf{n} >= 0$, we get

$$\begin{aligned}
\kappa_n &= < T', \mathbf{n} >=< \mathbf{x}_{\alpha\beta}, \mathbf{n} > \frac{du^\alpha}{ds}\frac{du^\beta}{ds} \\
&= \frac{b_{\alpha\beta}du^\alpha du^\beta}{g_{\alpha\beta}du^\alpha du^\beta}, \qquad (4.17)
\end{aligned}$$

where

$$b_{\alpha\beta} =< \mathbf{x}_{\alpha\beta}, \mathbf{n} > \qquad (4.18)$$

4.9 Definition The expression

$$II = b_{\alpha\beta} du^\alpha \otimes du^\beta \qquad (4.19)$$

is called the **second fundamental form** .

4.10 Proposition The second fundamental form is symmetric.
Proof: In the classical formulation of the second fundamental form the proof is trivial. We have $b_{\alpha\beta} = b_{\beta\alpha}$ since for a C^∞ patch $\mathbf{x}(u^\alpha)$, we have $\mathbf{x}_{\alpha\beta} = \mathbf{x}_{\beta\alpha}$ because the partial derivatives commute.
We will denote the coefficients of the second fundamental form by

$$\begin{aligned} e &= b_{11} =< \mathbf{x}_{uu}, \mathbf{n} > \\ f &= b_{12} =< \mathbf{x}_{uv}, \mathbf{n} > \\ &= b_{21} =< \mathbf{x}_{vu}, \mathbf{n} > \\ g &= b_{22} =< \mathbf{x}_{vv}, \mathbf{n} > \end{aligned}$$

, so that equation (4.19) can be written as

$$II = edu^2 + 2fdudv + gdv^2 \tag{4.20}$$

and equation (4.17) as

$$\kappa_n = \frac{II}{I} = \frac{Edu^2 + 2Fdudv + Gdv^2}{edu^2 + 2fdudv + gdv^2} \tag{4.21}$$

We would also like to point out that just as the first fundamental form can be represented as

$$I =< d\mathbf{x}, d\mathbf{x} >,$$

so can we represent the second fundamental form as

$$II = - < d\mathbf{x}, d\mathbf{n} >$$

To see this it suffices to note that differentiation of the identity $< \mathbf{x}_\alpha, \mathbf{n} >= 0$ implies that

$$< \mathbf{x}_{\alpha\beta}, \mathbf{n} >= - < \mathbf{x}_\alpha, \mathbf{n}_\beta > .$$

Therefore,

$$\begin{aligned} < d\mathbf{x}, d\mathbf{n} > &= < \mathbf{x}_\alpha du^\alpha, \mathbf{n}_\beta du^\beta > \\ &= < \mathbf{x}_\alpha du^\alpha, \mathbf{n}_\beta du^\beta > \\ &= < \mathbf{x}_\alpha, \mathbf{n}_\beta > du^\alpha du^\beta \\ &= - < \mathbf{x}_{\alpha\beta}, \mathbf{n} > du^\alpha du^\beta \\ &= -II \end{aligned}$$

From a computational point a view, a more useful formula for the coefficients of the second fundamental formula can be derived by first applying the classical vector identity

$$(A \times B) \cdot (C \times D) = \begin{vmatrix} A \cdot C & A \cdot D \\ B \cdot C & B \cdot D \end{vmatrix} \tag{4.22}$$

to compute

$$\begin{aligned} \|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= (\mathbf{x}_u \times \mathbf{x}_v) \cdot (\mathbf{x}_u \times \mathbf{x}_v) \\ &= det \begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v \end{bmatrix} \\ &= EG - F^2 \end{aligned} \tag{4.23}$$

Consequently, the normal vector can be written as

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{EG - F^2}}$$

Thus, we can write the coefficients $b_{\alpha\beta}$ directly as triple products involving derivatives of $(\mathbf{x})$. The expressions for these coefficients are

$$\begin{aligned} e &= \frac{(\mathbf{x}_u \mathbf{x}_u \mathbf{x}_{uu})}{\sqrt{EG - F^2}} \\ f &= \frac{(\mathbf{x}_u \mathbf{x}_v \mathbf{x}_{uv})}{\sqrt{EG - F^2}} \\ g &= \frac{(\mathbf{x}_v \mathbf{x}_v \mathbf{x}_{vv})}{\sqrt{EG - F^2}} \end{aligned} \tag{4.24}$$

The first fundamental form on a surface measures the (square) of the distance between two infinitesimally separated points. There is a similar interpretation of the second fundamental form as we show below. The second fundamental form measures the distance from a point on the surface to the tangent plane at a second infinitesimally separated point. To see this simple geometrical interpretation, consider a point $\mathbf{x}_0 = \mathbf{x}(u_0^\alpha) \in \mathcal{M}$ and a nearby point $\mathbf{x}(u_0^\alpha + du^\alpha)$. Expanding on a Taylor series, we get

$$\mathbf{x}(u_0^\alpha + du^\alpha) = \mathbf{x}_0 + (\mathbf{x}_0)_\alpha du^\alpha + \frac{1}{2}(\mathbf{x}_0)_{\alpha\beta} du^\alpha du^\beta + \ldots$$

We recall that the distance formula from a point $\mathbf{x}$ to a plane which contains $\mathbf{x}_0$ is just the scalar projection of $(\mathbf{x} - \mathbf{x}_0)$ onto the normal. Since the normal to the plane at $\mathbf{x}_0$ is the same as the unit normal to the surface and $< \mathbf{x}_\alpha, \mathbf{n} >= 0$, we find that the distance D is

$$\begin{aligned} D &= < \mathbf{x} - \mathbf{x}_0, \mathbf{n} > \\ &= \frac{1}{2} < (\mathbf{x}_0)_{\alpha\beta}, \mathbf{n} > du^\alpha du^\beta \\ &= \frac{1}{2} II_0 \end{aligned}$$

The first fundamental form (or rather, its determinant) also appears in calculus in the context of calculating the area of a parametrized surface. the reason is that if one considers an infinitesimal parallelogram subtended by the vectors $\mathbf{x}_u du$ and $\mathbf{x}_v dv$, then the differential of surface area is given by the length of the cross product of these two infinitesimal tangent vectors. That is

$$\begin{aligned} dS &= \|\mathbf{x}_u \times \mathbf{x}_v\| \, dudv \\ S &= \int\int \sqrt{EG - F^2} \, dudv \end{aligned}$$

The second fundamental form contains information about the shape of the surface at a point. For example, the discussion above indicates that if $b = |b_{\alpha\beta}| = eg - f^2 > 0$ then all the neighboring points lie on the same side of the tangent plane, and hence, the surface is concave in one direction. If at a point on a surface $b > 0$, the point is called an elliptic point, if $b < 0$, the point is called hyperbolic or a saddle point, and if $b = 0$, the point is called parabolic.

4.4 Curvature

Curvature and all related questions which surround curvature, constitute the central object of study in differential geometry. One would like to be able to answer questions such as, what quantities remain invariant as one surface is smoothly changed into another? There is certainly something intrinsically different from a cone, which we can construct from a flat piece of paper and a sphere which we can not. What is it that makes these two surfaces so different? How does one calculate the shortest path between two objects when the path is constrained to be on a surface?

These and many other questions of similar type can be quantitatively answered through the study curvature. We cannot overstate the great importance of this subject; perhaps it suffices to say that without a clear understanding of curvature, there would not be a general theory of relativity, no concept of black holes, and even more disastrous, no Star Trek.

The study of curvature of a hypersurface in $\mathbf{R}^n$ (a surface of dimension $n-1$) begins by trying to understand the covariant derivative of the normal to the surface. The reason is simple. If the normal to a surface is constant, then the surface is a flat hyperplane. Thus, it is variations in the normal that indicate the presence of curvature. For simplicity, we constrain our discussion to surfaces in $\mathbf{R}^3$, but the formalism we use is applicable to any dimension. We will also introduce in this section the modern version of the second fundamental form

4.11 Definition Let X be a vector field on a surface M in $\mathbf{R}^3$, and let N be the normal vector. The map L given by

$$LX = -\overline{\nabla}_X N \tag{4.25}$$

is called the **Gauss map**.

In this definition we will be careful to differentiate between operators which live on the surface and operators which live in the ambient space. We will adopt the convention of overlining objects which live in the ambient space, the operator $\overline{\nabla}$ above being an example of one such object. The Gauss map is clearly a good place to start, since it is the rate of change of the normal in an arbitrary direction which we wish to quantify.

4.12 Definition The **Lie bracket** $[X, Y]$ of two vector fields X and Y on a surface $\mathcal{M}$ is defined as the commutator

$$[X, Y] = XY = YX, \tag{4.26}$$

meaning that if f is a function on $\mathcal{M}$ then $[X, Y]f = X(Y(f)) - Y(X(f))$.

4.13 Proposition The Lie bracket of two vectors $X, Y \in T(calM)$ is another vector in $T(\mathcal{M})$.
Proof: If suffices to prove that the bracket is a linear derivation on the space of C^∞ functions. Consider vectors $X, Y, Z \in T(calM)$ and smooth functions f, g in $\mathcal{M}$. Then

$$\begin{aligned}
[X, Y+Z](f) &= X((Y+Z)(f)) - (Y+Z)X(f) \\
&= X(Y(f)+Z(f)) - Y(X(f)) - Z(X(f)) \\
&= X(Y(f)) - Y(X(f)) + X(Z(f)) - Z(X(f)) \\
&= [X, Y](f) + [X, Z](f),
\end{aligned}$$

and

$$\begin{aligned}
[X+Y, Z](f) &= (X+Y)(Z(f)) - Z((X+Y)(f)) \\
&= X(Z(f)) + Y(Z(f)) - Z(X(f)) - Z(Y(f)) \\
&= X(Z(f)) - Z(X(f)) + Y(Z(f)) - Z(Y(f)) \\
&= [X, Z](f) + [Y, Z](f),
\end{aligned}$$

so the bracket is linear on each slot. Furthermore

$$\begin{aligned}
[X, Y](f+g) &= X(Y(f+g)) - Y(X(f+g)) \\
&= X(Y(f)+Y(g)) - Y(X(f)+X(g)) \\
&= X(Y(f)) - Y(X(f)) + X(Y(g)) - Y(X(g)) \\
&= [X, Y](f) + [X, Y](g),
\end{aligned}$$

$$\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X[fY(g) + gY(f)] - Y[fX(g) + gX(f)] \\
&= X(f)Y(g) + fX(Y(g)) + X(g)Y(f) + gX(Y(f)) \\
&\quad -Y(f)X(g) - f(Y(X(g)) - Y(g)X(f) - gY(X(f)) \\
&= f[X(Y(g)) - (Y(X(g))] + g[X(Y(f)) - Y(X(f))] \\
&= f[X,Y](g) + g[X,Y](f),
\end{aligned}$$

so that the bracket acts as linear derivation of functions.

4.14 Proposition The Gauss map is a linear transformation on $T(\mathcal{M})$.
Proof: Linearity follows from the linearity of $\overline{\nabla}$, so it suffices to show that $L : X \longrightarrow LX$ maps $X \in T(\mathcal{M})$ to a vector $LX \in T(\mathcal{M})$. Since N is the unit normal to the surface, $< N, N >= 1$, so that any derivative of $< N, N >$ is 0. Assuming that the connection is compatible with the metric,

$$\begin{aligned}
\overline{\nabla}_X < N, N > &= < \overline{\nabla}_X N, > + < N, \overline{\nabla}_X N > \\
&= 2 < \overline{\nabla}_X N, N > \\
&= 2 < LX, N >= 0.
\end{aligned}$$

Therefore, LX is orthogonal to N, and hence it lies in $T(\mathcal{M})$.

We recall at this point that the in the previous section we gave two equivalent definitions $< d\mathbf{x}, d\mathbf{x} >$, and $< X, Y >$) of the first fundamental form. We will now do the same for the second fundamental form.

4.15 Definition The **second fundamental form** is the bilinear map

$$II(X,Y) =< LX, Y > \tag{4.27}$$

4.16 Remark It should be noted that the two definitions of the second fundamental form are consistent. This is easy to see if one chooses X to have components $\mathbf{x}_\alpha$ and Y to have components $\mathbf{x}_\beta$. With these choices, LX has components $-\mathbf{n}_a$ and $II(X,Y)$ becomes $b_{\alpha\beta} = - < \mathbf{x}_\alpha, \mathbf{n}_\beta >$ We also note that there is third fundamental form defined by

$$III(X,Y) =< LX, LY >=< L^2X, Y > \tag{4.28}$$

In classical notation, the third fundamental form would be denoted by $< d\mathbf{n}, d\mathbf{n} >$. As one would expect, the third fundamental form contains third order Taylor series information about the surface. We will not treat $III(X,Y)$ in much detail in this work.

4.17 Definition The **torsion** of a connection $\overline{\nabla}$ is the operator T such that $\forall X, Y$

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \tag{4.29}$$

A connection is called **torsion free** if $T(X,Y) = 0$. In this case,

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X,Y].$$

We will say much more later about the torsion and the importance of torsion free connections. For the time being, it suffices to assume that for the rest of this section, all connections are torsion free. Using this assumption, it is possible to prove the following very important theorem.

4.18 Theorem The Gauss map is a self adjoint operator on $T\mathcal{M}$.
Proof: We have already shown that $L : T\mathcal{M} \longrightarrow T\mathcal{M}$ is a linear map. Recall that an operator L on a linear space is self adjoint if $< LX, Y > = < X, LY >$, so that the theorem is equivalent to proving that that the second fundamental for is symmetric ($II[X, Y] = II[Y, X]$). Computing the difference of these two quantities, we get

$$\begin{aligned} II[X,Y] - II[Y,X] &= < LX, Y > - < LY, X > \\ &= < \overline{\nabla}_X N, Y > - < \overline{\nabla}_Y N, X > . \end{aligned}$$

Since $< X, N > = < Y, N > = 0$, and the connection is compatible with the metric, we know that

$$\begin{aligned} < \overline{\nabla}_X N, Y > &= - < N, \overline{\nabla}_X Y > \\ < \overline{\nabla}_Y N, X > &= - < N, \overline{\nabla}_Y X >, \end{aligned}$$

hence,

$$\begin{aligned} II[X,Y] - II[Y,X] &= < N, \overline{\nabla}_Y X > - < N, \overline{\nabla}_X Y >, \\ &= < N, \overline{\nabla}_Y X - \overline{\nabla}_X Y > \\ &= < N, [X,Y] > \\ &= 0 \qquad (\text{iff } [X,Y] \in T(\mathcal{M})) \end{aligned}$$

One of the most important topics in an introductory course linear algebra deals with the spectrum of self adjoint operators. The main result in this area states that if one considers the eigenvalue equation

$$LX = \kappa X \tag{4.30}$$

then the eigenvalues are always real and eigenvectors corresponding to different eigenvalues are orthogonal. In the current situation, the vector spaces in question are the tangent spaces at each point of a surface in $\mathbf{R}^3$, so the dimension is 2. Hence, we expect two eigenvalues and two eigenvectors

$$LX_1 = \kappa_1 X_1 \tag{4.31}$$

$$LX_2 = \kappa_1 X_2. \tag{4.32}$$

4.19 Definition The eigenvalues κ_1 and κ_2 of the Gauss map L are called the **principal curvatures** and the eigenvectors X_1 and X_2 are called the **principal directions**.

Several possible situations may occur depending on the classification of the eigenvalues at each point $\mathbf{p}$ on the surface:

1. If $\kappa_1 \neq \kappa_2$ and both eigenvalues are positive, then $\mathbf{p}$ is called an elliptic point
2. If $\kappa_1\kappa_2 < 0$, then $\mathbf{p}$ is called a hyperbolic point.
3. If $\kappa_1 = \kappa_2 \neq 0$, then $\mathbf{p}$ is called an umbilic point.
4. if $\kappa_1\kappa_2 = 0$, then $\mathbf{p}$ is called a parabolic point

It is also well known from linear algebra, that the the determinant and the trace of a self adjoint operator are the only invariants under a adjoint (similarity) transformation. Clearly these invariants are important in the case of the operator L, and they deserve to be given special names.

4.20 Definition The determinant $K = \det(L)$ is called the **Gaussian curvature** of $\mathcal{M}$ and $H = (1/2)\text{Tr}(L)$ is called the **mean curvature**.

Since any self-adjoint operator is diagonalizable and in a diagonal basis, the matrix representing L is $\mathrm{diag}(\kappa_1, \kappa_2)$, if follows immediately that

$$\begin{aligned} K &= \kappa_1\kappa_2 \\ H &= \frac{1}{2}(\kappa_1 + \kappa_2) \end{aligned} \tag{4.33}$$

4.21 Proposition Let X and Y be any linearly independent vectors in $T(\mathcal{M})$. Then

$$\begin{aligned} LX \times LY &= K(X \times Y) \\ (LX \times Y) + (X \times LY) &= 2H(X \times Y) \end{aligned} \tag{4.34}$$

Proof: Since $LX, LY \in T(\mathcal{M})$, they can be expresses as linear combinations of the basis vectors X and Y.

$$\begin{aligned} LX &= a_1X + b_1Y \\ LY &= a_2X + b_2Y. \end{aligned}$$

computing the cross product, we get

$$\begin{aligned} LX \times LY &= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} X \times Y \\ &= \det(L)(X \times Y). \end{aligned}$$

Similarly

$$\begin{aligned} (LX \times Y) + (X \times LY) &= (a_1 + b_2)(X \times Y) \\ &= \mathrm{Tr}(L)(X \times Y) \\ &= (2H)(X \times Y). \end{aligned}$$

4.22 Proposition

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ H &= \frac{1}{2}\frac{Eg - 2Ff + eG}{EG - F^2} \end{aligned} \tag{4.35}$$

Proof: Starting with equations (4.34) take the dot product of both sides with $X \times Y$ and use the vector identity (4.22). We immediately get

$$K = \frac{\begin{vmatrix} <LX, X> & <LX, Y> \\ <LY, X> & <LX, X> \end{vmatrix}}{\begin{vmatrix} <X, X> & <X, Y> \\ <Y, X> & <Y, Y> \end{vmatrix}}$$

$$2H = \frac{\begin{vmatrix} <LX, X> & <LX, Y> \\ <Y, X> & <Y, Y> \end{vmatrix} + \begin{vmatrix} <X, X> & <X, Y> \\ <LY, X> & <LY, Y> \end{vmatrix}}{\begin{vmatrix} <X, X> & <X, Y> \\ <Y, X> & <Y, Y> \end{vmatrix}}$$

The result follows by taking $X = \mathbf{x}_u$ and $Y = \mathbf{x}_v$

4.23 Theorem (Euler) Let X_1 and X_2 be unit eigenvectors of L and let $X = (\cos\theta)X_1 + (\sin\theta)X_2$. Then

$$II(X,X) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \tag{4.36}$$

Proof: Easy. Just compute $II(X,X) = < LX, X >$, using the fact the $LX_1 = \kappa_1 X_1$, $LX_2 = \kappa_2 X_2$, and noting that the eigenvectors are orthogonal. We get,

$$\begin{aligned} < LX, X > &= < (\cos\theta)\kappa_1 X_1 + (\sin\theta)\kappa_2 X_2, (\cos\theta)X_1 + (\sin\theta)X_2 > \\ &= \kappa_1 \cos^2\theta < X_1, X_1 > + \kappa_2 \sin^2\theta < X_2, X_2 > \\ &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta. \end{aligned}$$