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Mathematics Resource Part I of III: Algebra

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FOR

MY TWO COACHES (MRS. REEDER, MS. MARBLE, AND MRS. STRINGHAM) FOR THEIR PATIENCE, UNDERSTANDING, KNOWLEDGE, AND PERSPECTIVE

DEMIDEC

RESOURCES AND EXAMS

ALGEBRA

A LITTLE ON THE NATURE OF NUMBERS

Real Number	**Additive Inverse**	**Multiplicative Inverse**	**Subtraction**
Division	**Cancellation Law**	**Negative**	**Distributive Property**
Commutative Property	**Associative Property**	**Additive Identity**	**Multiplicative Identity**

When you think about math, what comes to your mind? Numbers. Numbers make the world go round; they can be used to express distances and amounts; they make up your phone number and zip code, and they mark the passage of time. The concepts connected with numbers have been around for ages. A **real number** is any number than can exist on the number line. At this point in your schooling, you are likely to have already come across the number line; it is usually drawn as a horizontal line with a mark representing zero. Any point on the line can represent a specific real number.[1]

One of the easiest and most obvious ways to classify numbers is as either positive or **negative**. What does it mean for a number to be negative? Well, first of all, it is graphed to the left of the zero mark on a horizontal number line, but there's more. A negative signifies the opposite of whatever is negated. For example, to say that I walked east 50 miles would be mathematically equivalent to saying that I walked west negative 50 miles.[2] I could also say that having a bank balance of -\$41.90 is the same as being \$41.90 in debt. The negative in mathematics represents a logical opposite.

When two numbers are *added*, their values combine. When two numbers are multiplied, we perform repeated (or multiple) additions.

Examples:

$3 + 5 = 8$ $-11 + 9 = -2$ $19 - 2 = 17$ $-3 + 91 = 88$ $12 - 15 = -3$

$3 \times 5 = 5 + 5 + 5 = 15$ $4 \times 2 = 2 + 2 + 2 + 2 = 8$

$5 \times 1 = 1 + 1 + 1 + 1 + 1 = 5$ $2 \times 4 = 4 + 4 = 8$

Here, I'm just rehashing things with which most of you readers are probably already acquainted.[3] I know of very few high school students (and even fewer decathletes) who have trouble with basic addition and multiplication of real numbers. Sometimes, negatives complicate the fray a bit, but for a brief review, you should know the negation rules for multiplication and division.

[1] It's possible you haven't yet come across non-real numbers. I wouldn't worry about it. Non-real numbers enter the picture when you take the square root of negatives, and they shouldn't be your concern this decathlon season.

[2] Um… I wouldn't recommend actually saying something like this on a regular basis to ordinary people. I just wouldn't. Trust me on this one.

[3] In fact, this resource is going to operate under the assumption that decathletes already have experience with much of this year's algebra curriculum. I'm not going to go into detail about the mechanics of arithmetic. I'm also, rather presumptuously, going to use ×, •, and ( ) interchangeably to indicate multiplication.

- Negative × Negative = Positive
- Negative × Positive = Negative
- Positive × Negative = Negative
- Negative ÷ Negative = Positive
- Negative ÷ Positive = Negative
- Positive ÷ Negative = Negative

Make a note that these sign patterns are the same for both multiplication and division; we'll talk more about that in just a quick sec. Also, notice that I didn't list addition and subtraction properties of negative numbers. When something is negative, it means we go leftward on the number line, while positives take us rightward. When you add and subtract positives with negatives, the sign of the answer will have the same sign as the "bigger" number.

Note a few more examples here:

$5 + -5 = 0$ $\quad 2 \times \frac{1}{2} = 1$

$3 + -3 = 0$ $\quad 5 \times \frac{1}{5} = 1$

In the examples above, we see two instances of two numbers adding to 0 and two instances of two numbers multiplying to a product of 1. If you look closely, there is consistency here. The **additive inverse** (or the *opposite*) of any number "x" is denoted by "-x." The **multiplicative inverse** (or the *reciprocal*) of any number "y" is written "$\frac{1}{y}$." A number and its additive inverse sum to zero; a number and its multiplicative inverse multiply to one.

Additive Inverse of a: $-a$

$a + (-a) = 0$

Multiplicative inverse of a: $\frac{1}{a}$

$a \times \frac{1}{a} = 1$

Note a few more examples here:

$-3 + 0 = -3$ $\quad 12 \times 1 = 12$

$9 + 0 = 9$ $\quad -8 \times 1 = -8$

In these four examples, we see two instances of the addition of 0 and two instances of multiplication by 1. The operations "adding 0" and "multiplying by 1" produce results identical to the original numbers, and thus we can name two mathematical identities.

The Additive Identity:

$a + 0 = a$

The Multiplicative Identity:

$a \times 1 = a$

0 is known as the "additive identity element," and 1 is known as the "multiplicative identity element." With identities and inverses in mind, we can continue with our discussion of algebra. To say "x + -x = 0" is the same as "x – x = 0." This may sound weird to say at first, but it is one of the closely guarded secrets of mathematics that subtraction and division, as separate operations, do not really exist. Youngsters are trained to perform simple procedures that they call subtract and divide, but from a mature, sophisticated, *mathematical* point of view, those operations are nothing more than special cases of addition and multiplication.

Definition of Subtraction:
$x - y = x + (-y)$
Definition of Division:
$x \div y = x \bullet \frac{1}{y}$

In addition to knowing these formal definitions for subtraction and division, the astute decathlete should be (and probably already is) familiar with several properties of the real numbers. These include the **Commutative Properties**, the **Associative Properties**, and the **Distributive Properties**.

Commutative Property of Addition:	$m + n = n + m$
Commutative Property of Multiplication:	$m \bullet n = m \bullet n$
Associative Property of Addition:	$a + (b + c) = (a + b) + c$
Associative Property of Multiplication:	$a \bullet (b \bullet c) = (a \bullet b) \bullet c$
Distributive Property:	$a \bullet (b + c) = a \bullet b + a \bullet c$

Example:
Arbitrarily pick some real numbers and verify the distributive property.

Solution:
I'll choose 3, 5, and 7, for a, b, and c, respectively.
Distributive Property: $3 \bullet (5 + 7) = 3 \bullet 5 + 3 \bullet 7$. Can we verify this? The left side of the equation gives $3 \bullet (5 + 7) = 3 \bullet 12 = 36$. The right side of the equation gives $3 \bullet 5 + 3 \bullet 7 = 15 + 21 = 36$. The Distributive Property holds.

Be wary; sometimes, confused students have conceptual problems with the Distributive Property. I have on occasion seen people write that $a + (b \bullet c) = a + b \bullet a + c$. Such a thing is wrong. Remember that multiplication distributes over addition, not vice versa.

This is also a good time to discuss the algebraic order of operations. The example above assumes an elementary knowledge that operations grouped in parentheses are performed first. The official mathematical order of operations is Parentheses/Groupings, Exponents[4], Multiplication/Division, Addition/Subtraction. In many pre-algebra and algebra classes, a common mnemonic device for this is "**P**lease **e**xcuse **m**y **d**ear **A**unt **S**ally."

A brief example is now obligatory to expand on the order of operations.

Example:
$$\frac{-3^2 - (-4 + (-2)^4)}{1 + 2 \times 3} + 3$$

Solution:
This may seem a little extreme as a first example, but it is fairly simple if approached systematically. Remember, the top and bottom (that's *numerator* and *denominator* for you terminology buffs) of a fraction should generally be evaluated separately and first; a giant fraction bar is a form of parentheses, a grouping symbol. On the top, we find two sets of parentheses, and start with the inside one, so -2 is our starting point. The exponent comes first, so we evaluate $(-2)^4 = (-2)(-2)(-2)(-2) = 16$. Then, substituting gives -4 + 16 = 12. We

[4] An exponent, if you do not know, is a small superscript that indicates "the number that I'm above is multiplied by itself a number of times equal to me." If it helps, imagine the exponent saying this in a cute pair of sunglasses. For example, $3^4 = 3 \times 3 \times 3 \times 3 = 81$. 4 is the exponent. Come to think of it, exponents look a lot like footnote references. - Craig

are not done, but the whole expression reduces to the considerably simpler $\frac{-3^2-12}{1+2\times 3}+3$.

The first bit in the numerator will cause the most misery in this expression, as many people make this very common error: $-3^2 = (-3)(-3) = 9$. DON'T DO THIS! By our standard order of operations, the exponent must be evaluated first. It is often convenient to think of a negative sign as a $(-1)\bullet$, rather than a subtraction. By order of operations, negatives are evaluated with multiplication. The correct evaluation of the numerator is -21:

$-3^2 - 12 =$
$(-1) \times 3^2 - 12 =$
$(-1) \times 9 - 12 =$
$-9 - 12 =$
-21

Once this is done, we turn our attention to the fairly straightforward denominator. We take order of operations into account here.

$1 + 2 \times 3 = 1 + 6 = 7$. Now we put everything back into the original expression, and matters seem far simpler: $\frac{-21}{7}+3=-3+3=0$. I guess you could say we did all of that work to get nothing for our answer. Hah! Never forget the difference between the forms $(-x)^y$ and $-x^y$!

The last of the algebra basics to be discussed is the **cancellation law**. The cancellation law in its abstract form can look quite intimidating.

Cancellation Law:

$\frac{ab}{ac}=\frac{b}{c}$ as long as $a \neq 0$ and $c \neq 0$

This little formula can be quite intimidating, but the cancellation law in layman's terms says that anything divided by itself is 1 and can be "cancelled out." You've probably been using this law for quite some time, possibly without even realizing it, to simplify fractions. You know of course that $\frac{8}{12}=\frac{2}{3}$, but you may have become so familiar with the practice that you're not even aware of the cancellation law operating "behind the scenes". Observe:

$$\frac{8}{12}=\frac{2\cdot\cancel{4}}{3\cdot\cancel{4}}=\frac{2}{3}$$

The numerator and denominator are written as products, and a common factor of 4 is "cancelled out." Don't think that simplifying fractions is the only application for the cancellation law, though. It's that very law, albeit applied in reverse, that allows us to produce common denominators.

Examples:
Perform the following operations and simplify the answers.

a) $\frac{1}{2}+\frac{1}{6}=$

b) $\frac{1}{3}+\frac{1}{2}\bullet\frac{5}{6}=$

c) $\frac{2}{3}\div\frac{5}{3}=$

d) $\frac{1}{4}-\frac{1}{6}\div\frac{1}{3}=$

Solutions:

a) The addition of fractions requires a common denominator. Remember that getting a common denominator requires nothing more than applying the cancellation law in reverse; that is, multiplying by a cleverly chosen form of 1. For this problem, we'll multiply the first fraction by 1 in the form of $\frac{3}{3}$.

$$\frac{1}{2}+\frac{1}{6}=$$
$$\frac{1\cdot 3}{2\cdot 3}+\frac{1}{6}=$$
$$\frac{3}{6}+\frac{1}{6}=$$
$$\frac{4}{6}=\frac{2\cdot 2}{3\cdot 2}=\frac{2}{3}$$

b) Remember that in the order of operations, multiplication is always done before addition (unless there are parentheses involved).

$$\frac{1}{3}+\frac{1}{2}\bullet\frac{5}{6}=$$
$$\frac{1}{3}+\frac{1\cdot 5}{2\cdot 6}=$$
$$\frac{1}{3}+\frac{5}{12}=$$
$$\frac{1\cdot 4}{3\cdot 4}+\frac{5}{12}=$$
$$\frac{4}{12}+\frac{5}{12}=$$
$$\frac{9}{12}=\frac{3\cdot 3}{4\cdot 3}=\frac{3}{4}$$

c) To deal with division of fractions, you'll need to remember that according to the algebraic definition of division, there is no such thing as division at all. Division by x is simply multiplication by the multiplicative inverse, or reciprocal, of x.

$$\frac{2}{3} \div \frac{5}{3} =$$
$$\frac{2}{3} \times \frac{3}{5} =$$
$$\frac{6}{15} = \frac{2}{5}$$

d) Remember again the order of operations. The division here must occur before the subtraction can be done.

$$\frac{1}{4} - \frac{1}{6} \div \frac{1}{3} =$$
$$\frac{1}{4} - \frac{1}{6} \times 3 =$$
$$\frac{1}{4} - \frac{3}{6} =$$
$$\frac{1}{4} - \frac{1 \cdot 2}{2 \cdot 2} =$$
$$\frac{1}{4} - \frac{2}{4} =$$
$$-\frac{1}{4}$$

Perhaps you are already well-versed in the rules for and procedures involved in the arithmetic of fractions. If so, these examples and all of the steps displayed probably seemed unnecessary and extravagant. Soon, however, in a discussion of rational expressions, we will refer back to these examples and use them as a models for more complicated mathematics. Until then, we move on.

EEP: EXPRESSIONS, EQUATIONS, AND POLYNOMIALS

Expression	**Equation**	**Monomial**	**Polynomial**
Equivalent Equations	**Constant**	**Variable**	

In the previous section, when working through the early example to verify the distributive property, I wrote down, using numbers, 3 • (5 + 7) = 3 • 5 + 3 • 7. When the property was originally written, however, it was listed as “a • (b + c) = a • b + a • c.” What is the difference between these two listings of the distributive property? It should be obvious.[5] The property was originally listed using letters while the example instance used numbers. A **variable** in algebra is a symbol (almost always a letter, occasionally Greek) that represents a number or group of numbers the specific value of which is not known. A **constant** is a symbol that represents only one value. In other words, a variable *represents* some number(s) and a constant *is* some number.

Example:
Make a short list of possible variable names, and make another list of constants.

[5] If it's not obvious, then <insert your own witty insult here>.

Solution:
Variables: x, y, z, a, n, θ, ϕ
Constants: 1, 2, 3, -12.90, 7.0001, $\sqrt{5}$, $e \approx 2.7183$, $\pi \approx 3.1416$

These variables and constants together make up the "nouns" of algebra. Any string of variables and/or constants connected by algebraic operators (the "action verbs" of algebra) that can represent a value is an **expression**. Possible expressions include $x+12$, $\frac{413}{2}\sqrt[3]{10}$, y, $\sqrt{200n}$, $3x^3 - 4x - 6$, and 1. Notice that single variables as well as lone numbers qualify as expressions.

Subsequently, an **equation** in algebra is a statement that two expressions have the same value; the verb is the "=" symbol and is read "equals." Conventionally, everyone thinks of algebra as a math that involves solving equations; what does it mean, though, to "solve" an equation? As regards equations with one variable – if an equation states that two expressions are, without fail, equal to each other with only one distinct variable in common, then a person doing mathematics can explicitly solve for all values of that variable that can make the equation true. Let's look at some sample equations.

Sample Equation #1:
$14 + 6 = 4 \times 5$

Sample Equation #2:
$x - 12 = 3x + 4$

Sample Equation #3:
$x^4 - 3x^3 + 2x^2 + 7x + 9 = x^4 - 3x^3 + 2x^2 + 7x + 9$

Sample Equation #4:
$x^2 + 3x = -10$

Sample Equation #5:
$x + 4 = x - 2$

What can we say about these equations? Well, the first one is obviously true. When simplified, it gives us that 20 = 20. The other four equations are bit harder to assess—unless we assign a particular value to x, we cannot say whether the equations are true or false statements. What we can do though, and this is the part of algebra that the average person is most familiar with, is solve the equations to find the value(s) of x that result in true statements.
In order to do this, we transform each equation into **equivalent equations**. Equivalent equations are equations that have the same "meaning" as each other; in math terms, we say that the equations have the same solution set. For example, I do not need to tell you how to solve an equation for x such as "x + 5 = 11." Common sense is just fine. What value, when five is added to it, gives eleven? The answer is six. To say "x = 6" is an equivalent equation to the one earlier. It would also be an equivalent equation to say that "x – 1 = 5."

Transformations are mathematical operations that can produce equivalent equations. To go from the first equation, "x + 5 = 11," to the second equation, "x = 6," what was done? The value of -5 was added to each side (remember, we could also say that 5 was subtracted from each side – it has the same meaning). An elementary school math teacher introducing my class to the concept of equations once told me, "Think of an equation as a scale saying that two things weigh exactly the same. If you could do something to that scale that keeps the sides weighing the same, then you can do it to an equation." By far the two most common transformations that equations undergo are (1) the addition of an identical value to both sides and (2) the multiplication of an identical value to both sides. I won't bother listing subtraction or division because I'm a bit stuck on the idea that they are just special forms of addition and multiplication. With that in mind, let's attempt to *transform* a somewhat complicated equation in order to solve for x.

Example:

Solve for x in the equation $\frac{3x-4}{-4} = x - \frac{1}{2}$

Solution:

Since x found on both sides of the equation, there will have to be steps taken to *isolate* the variable on a single side of the equation. Here is the list of equivalent equations, along with the steps required to produce each.

$\frac{3x-4}{-4} = x - \frac{1}{2}$

$3x – 4 = -4\ (x – \frac{1}{2})$ ← multiply each side by the multiplicative inverse of $-\frac{1}{4}$

$3x – 4 = -4x + 2$ ← apply the distributive property to the right-side expression

$3x = -4x + 6$ ← add the additive inverse of -4 to each side

$7x = 6$ ← add the additive inverse of -4x to each side

$x = \frac{6}{7}$ ← multiply each side by the multiplicative inverse of 7

All six of the lines listed above are equivalent equations. Notice that the third line involved transforming only one side of the equation (with the distributive property), but that all of the other transformations were accomplished by either adding an additive inverse to both sides or multiplying both sides by a multiplicative inverse. These transformations help eliminate the complexities around the variable and help solve the equation.

Getting back now to the idea of the expression, there are certain expressions that deserve special attention: **monomials** and **polynomials**. A *monomial* is any term like $3x^2$ or πn^4 that is the product of a constant and a variable raised to a nonnegative integral power.[6] A *polynomial* on the other hand, refers to any sum of monomials. In mathematician jargon, a polynomial is "Any expression that can be written in the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$, where each a_i is a constant and n is an integer." Some examples of polynomials of one variable are:

4
$2x + 6$
$3x^2 – 12x + 4$
$4x^{10} + x^9 + 41x^8 – 3x^7 – 6$

We need to point out a few things about these examples: first, notice the order in which the terms of each polynomial (we'll talk about that last one in just a second) fall. The term with the largest exponent is always written first and the remaining terms are arranged so that the exponents are in descending order. This arrangement is referred to as the *standard form* of the polynomial. While the commutative property of addition assures us that $–12x + 3 + 3x^2$ and $3x^2 – 12x + 4$ are equal, the non-standard version just doesn't appear in reputable mathematical writing.

All right, so if the terms of polynomials ought to be arranged in order of descending exponents, what about that long one up there? Shouldn't there be terms containing x^6, x^5, and so forth, in $4x^{10} + x^9 + 41x^8 – 3x^7 – 6$? Well, that's the second thing that we need to point out here: standard form does not require that the exponent decrease by exactly 1 with each successive term—$4x^{10} + x^9 + 41x^8 – 3x^7 – 6$ is a perfectly legitimate polynomial in spite of a few "missing" terms. (Some people like to think of those absent terms as simply invisible because their coefficients are 0.[7])

The last thing that needs to be mentioned about these examples is that last polynomial. Yes, that's right, 4 is a full-fledged polynomial even though it consists of but a single constant—monomials are special-case polynomials.

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[6] Even something like $3\pi x^2n^4$ is a monomial, but this year's official decathlon curriculum says that competition tests will only deal with monomials and polynomials in *one variable* and thus, so will we.

[7] They did not think; therefore, they were not.

What can we do with polynomials? Well, in higher math circles, polynomials form what is known as a *ring*. This is a fancy way of saying that any sum, difference, or product of polynomials will also be a polynomial.[8] The exact properties of rings, however, are not our concern in the least. What *is* important here is that we know *how* to find the sums and differences of polynomials. To do so, we identify all terms that contain the same variable(s) raised to the same power(s) and then we add (or subtract) the coefficients of those terms. The procedure is usually called *combining like terms*. Perhaps a few examples are in order.

Example:
Find the sum $(-x^3 + 4x^2 + 8x - 4) + (x^2 - 3x + 4)$.

Solution:
Because of the associative and commutative properties of addition, we can combine these terms in any order that we want. The best thing to do is rearrange the terms where we can see the like terms together.
$-x^3 + (4x^2 + x^2) + (8x + -3x) + (-4 + 4)$ is the result of grouping like terms together, and when we add the coefficients of the like terms we get $-x^3 + 5x^2 + 5x$.

Example:
Find the difference $(-x^3 + 4x^2 + 8x - 4) - (x^2 - 3x + 4)$.

Solution:
This looks strikingly similar to the previous example, except that now we are taking a difference of two polynomials. First, we will apply the definition of subtraction and the distributive property to "distribute the negative" over the parentheses and then add the result.
$(-x^3 + 4x^2 + 8x - 4) - (x^2 - 3x + 4) =$
$(-x^3 + 4x^2 + 8x - 4) + -1 \bullet (x^2 - 3x + 4) =$ ← definition of subtraction
$(-x^3 + 4x^2 + 8x - 4) + (-x^2 + 3x + -4) =$ ← distributive property
[From this point, we can apply the same technique used in the example above.]
$-x^3 + 4x^2 + -x^2 + 8x + 3x - 4 + -4 =$ ← commutative property
$-x^3 + 3x^2 + 11x - 8$

Several pages ago, five sample equations were displayed. Take a look now at equation #3. You should see that the same polynomial occurs on either side of the equal sign, and it should be evident that if we begin the solution process by adding additive inverses to each side, we end up pretty quickly with the equivalent equation "0 = 0". What does it mean when a statement involving a variable simplifies to an equation that is always true? It means that the original equation is true for any value of the variable—the solution to the equation is the set of all real numbers. No real number can possibly falsify the equation.

Look, too, at the 5th of those sample equations. If we attempt to solve $x + 4 = x - 2$ by adding the additive inverse of x to both sides of the equation, we'll be faced with the rather dubious statement of $4 = -2$. Not even in magical fairy lands can this be true. This 5th equation has no solution at all because no possible value of x can transform that false statement into a true one.

AN EQUAL UNEQUAL

Inequality **Equivalent Inequality** **Absolute Value**

We have now discussed equations and "solving things" in some detail.[9] It is time to move on to other mathematical statements. "But what other mathematical statements ARE there besides saying that two things are equal?" you ask enthusiastically, eager to learn more math. Well, rather predictably, I respond that there are mathematical statements that two things are NOT equal, of

[8] Obligatory spiel about math: A mathematical ring must also meet some other requirements. If you're curious, feel free to consult a math major or professor at any university.
[9] I asked someone the relatively deep question once, "What exactly IS algebra?", to which I received the response, "Umm… solving things." Touché.

course. Any mathematical statement saying that the value of two expressions is not equal is an **inequality**. As luck (and maybe the math gods) would have it though, the terminology and techniques of inequalities are remarkably similar to their counterparts in the world of equations. The solution set of an inequality is the set of numbers that makes the inequality true, and **equivalent inequalities** are inequalities with the same solution sets for the variable(s) involved.

$\leq$	-	"less than or equal to" or "at most"
$\geq$	-	"greater than or equal to" or "at least"
$<$	-	"less than"
$>$	-	"greater than"
$\neq$	-	"not equal to"

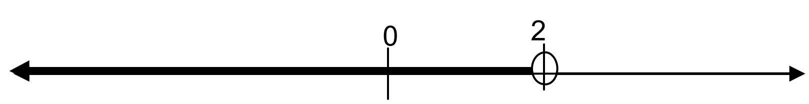
In the box above, there are five inequality symbols listed with their most common verbal equivalents. Every mathematical inequality will use one of the five symbols. Inequalities are solved in much the same way as equations; additive inverses are added and multiplicative inverses are multiplied until the variable is isolated and explicitly stated. The solution of an equation is generally a simpler more explicit equation—$x = 4$, for example—so it shouldn't surprise you in the least to learn that the solution of an inequality is generally a simpler inequality—something like $x > 5$—that gives all possible values of the variable.

If you are not familiar with the properties of inequalities, the verbal statements of the signs is almost enough to guide you. To write "$x \leq 7$" means that x can take any value less than or equal to 7. This inequality states that x could be 7, 0, 5.381, or even -1,000,000. The inequality "$x < 7$" is different from "$x \leq 7$" *only* in that *x* cannot be 7 exactly—with the exception of that one detail, the two inequalities $x \leq 7$ and $x < 7$ have the same solution set.

Example:
Solve $4x + 3 < 11$ for x.

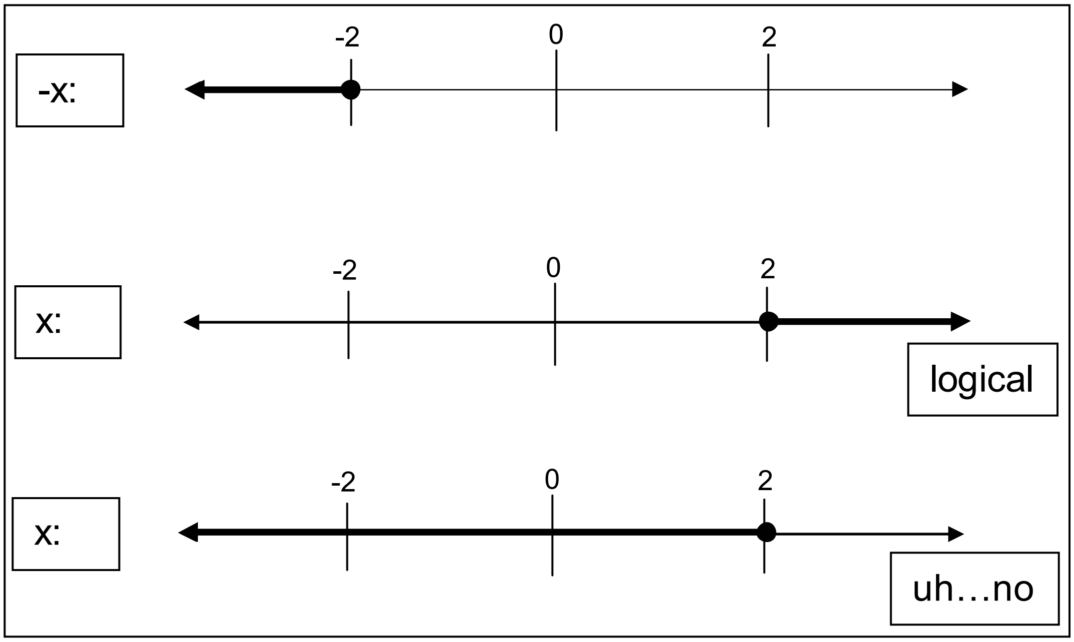
Solution:
$4x + 3 < 11$
$4x < 8$ ← add the additive inverse of 3 to each side
$x < 2$ ← multiply by the multiplicative inverse of 4 on each side

The solution for x in this inequality is $x < 2$, meaning that the variable x could take on any value less than (but not including) 2 and still create a true statement. On this number line, the shaded region represents the values that x could take.



The solutions to an inequality are often graphed on a horizontal number line for clearness. Here, the number line is shaded to the left, meaning that values less than two will satisfy the inequality. Also note the shaded line ending in the open circle, indicating that all values up to *but not including* 2 are valid solutions to this particular inequality. It might be intuitive that a closed circle would mean that all values less than *and including* 2 are correct.

It's been pointed out in words and by example that the procedure for solving inequalities is very similar to that used to solve equations. There is, however, one significant aspect to solving inequalities that the algebra enthusiast (or non-enthusiast, for that matter) must be acutely aware of: when both sides of an inequality are multiplied (or divided) by a negative number, the inequality symbol must "flip"—that is, the $\geq$ symbol must become $\leq$, and the $<$ symbol must become $>$. At first, this may seem to make no logical sense but if you remember that "to negate" is just another way to say "take the opposite," then it might start to make sense. If we have $-x \leq -2$ and we multiply both sides by -1, then we're looking for "the opposite" inequality. Is that $x \geq 2$ or $x \leq 2$? Look at the number lines…



Remember, the inequality sign ***must*** be flipped when a multiplication by a negative is introduced.

Example:
Solve $-2x + 4 \geq -7x - 16$ for x.

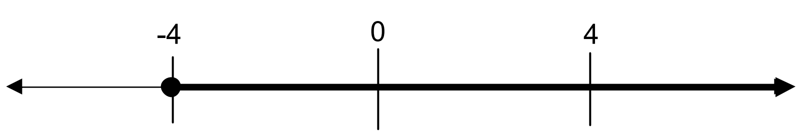
Solution:
We can solve this inequality by isolating x on either side of the equation. First, let's solve the inequality by isolating x on the left-hand side.

$-2x + 4 \geq -7x - 16$	
$5x + 4 \geq -16$	← adding the additive inverse of -7x to each side
$5x \geq -20$	← adding the additive inverse of 4 to each side
$x \geq -4$	← multiplying by the multiplicative inverse of 5 on each side

Now, let's try solving the inequality by isolating x on the right-hand side.

$-2x + 4 \geq -7x - 16$	
$4 \geq -5x - 16$	← adding the additive inverse of -2x to each side
$20 \geq -5x$	← adding the additive inverse of -16 to each side
$-4 \leq x$	← multiplying by the multiplicative inverse of -5 on each side, and flipping the inequality symbol
$x \geq -4$	← if we say that -4 is less than or equal to x, then that means that x is greater than or equal to -4

The solution is pictured below.



The algebra component of this year's math curriculum is to focus primarily on solving linear equations and inequalities. As long as you remember to add the same quantity to both sides, multiply by the same factor on both sides, and flip the inequality symbols when necessary, these test questions should pose no huge problem. There, is, however, one more topic concerning these single-variable equations/inequalities that should be addressed: absolute value.

Absolute value is a very unique mathematical operator. No matter what value it takes in, it spits out a positive value as a result. On paper, the absolute value of a quantity is represented by a pair of vertical lines surrounding that quantity.

Examples:
Find the following: |-2|, |4|, |-12.08|, $|-\frac{3}{5}|$, |12|, and |x|.

Solution:
Again, no matter what quantity the absolute value operator takes in, it gives a positive-valued result. This means that the five examples listed above take the values 2, 4, 12.08, $\frac{3}{5}$, and 12, respectively. The last example, the simplification of |x|, is a bit harder to write. We cannot simply write the answer as |x| = x because we do not know the value of x. If x were to equal -3, for example, we would have just asserted that |-3| = -3. The algebraic definition of absolute value is given below.

$\|x\| = x,$	if $x \geq 0$
$\|x\| = -x,$	if $x < 0$

"Whoa!" you say. "How is it possible that the absolute value of anything can be negative?" The answer is that it cannot. Look closely at that definition again and think "the opposite" when you see a negative sign. |x| = -x is only a true equation if x < 0. Try it with a few numbers. Input a positive number, and you get that positive number back. Input a negative number—you get that number's opposite. It works! Now that we have understanding concerning the properties of absolute value, we are left with the inevitable: solving equations and inequalities with absolute value. Comprehension comes here most easily with examples.

Example:
Solve |x| = 4 for all possible values of x.

Solution:
We want to know what numbers have an absolute value of 4. This is not difficult; the possibilities are either x = 4 or x = -4.

Example:
Solve |y| = 11 for all possible values of y.

Solution:
We now want to know what values have an absolute value of 11. This is not difficult, either; the possibilities are either y = -11 or y = 11.

These previous two examples are probably the easiest that absolute value equations can possibly get. Note that in both instances there are two possible solutions. This will be the case as long as the quantity within the absolute value bars equals some quantity greater than 0.

> | x | = c means that x = c or x = -c,
> as long as c > 0

This takes care of the simplest absolute value equations. What now about the slightly more complicated ones? Let's again inspect a few examples.

Example:
Solve $|z - 12| = 3$ for all possible values of z.

Solution:
This is only marginally more complicated than the previous examples. We know that the *quantity inside the absolute value bars* must be either 3 or -3 so we know that we have the two equations $z - 12 = -3$ or $z - 12 = 3$. From there, we can solve the equations individually to obtain $z = 9$ or $z = 15$.

Example:
Solve $|3a + \pi| = 14$ for all possible values of a.

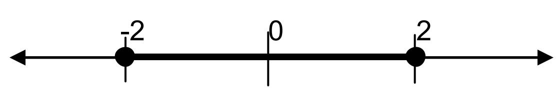
Solution:
This absolute value equation now is again becoming more complicated. We know that the quantity inside the absolute value bars, $3a + \pi$, must be equal to either -14 or 14 so we write the customary two equations and solve both.

$3a + \pi = -14$	or	$3a + \pi = 14$	
$3a = -14 - \pi$	or	$3a = 14 - \pi$	← add the additive inverse of π to each side
$a = \frac{-14 - \pi}{3}$	or	$a = \frac{14 - \pi}{3}$	← multiply each side by the mult. inverse of 3
$a \approx -5.714$	or	$a \approx 3.619$	← find the decimal approximations

These examples illustrate the concept of absolute value. Whatever quantity sits comfortably inside the absolute value bars must equal either the positive or the negative of the value that it is set equal to. Sadly, the official decathlon curriculum this year does not expect decathletes to solve equations concerning absolute value. Instead, it lists "solution of basic inequalities containing absolute value." Inequalities containing absolute value are a bit more complicated than equations but are still quite manageable. Much like absolute value equations, absolute value inequalities are probably best understood by examples.

Example:
Solve $|x| \leq 2$ for x.

Solution:
We start by examining the inequality, looking for some logical route to follow.[10] Perhaps if we start listing possible solutions to the equation, we can figure out the solution. Possible values of x that can make this a true equation are 1, 0, -1, 1.8, 1.201, -1.99, -0.3, 0.97, 2, and -1.41. Eureka! There is indeed a pattern. x will be any value between -2 and 2. In math language, this means that both $x \leq 2$ and $x \geq -2$. We might also just simplify our lives entirely by writing $-2 \leq x \leq 2$. One thing that is very important to note here is that many textbooks refer to absolute value as the distance from 0 on a number line. In that sense, the inequality itself says "x is no more than 2 units away from 0 on a number line."

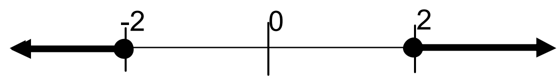


Example:
Solve $|y| \geq 2$ for y.

Solution:
It would make sense if the solution of this problem included all numbers that were not in the solution of the previous example. Possible values of y that can make this a true equation by having absolute value greater than 2 are 3, 10, 1400, -2.091, -5, -12, and -100. Essentially, the solution giving all possible y values is all numbers such that either $y \geq 2$ or $y \leq -2$.

[10] This is very important. Many people, when doing algebra, start blindly following procedures that have been programmed into them. Forgetting to *think* is a bad thing.

Considering our alternate definition of absolute value, the inequality reads “the distance of y to the origin is greater than or equal to 2.” No problem.



These two examples illustrate the general solutions to absolute value inequalities, stated concisely in the box below.

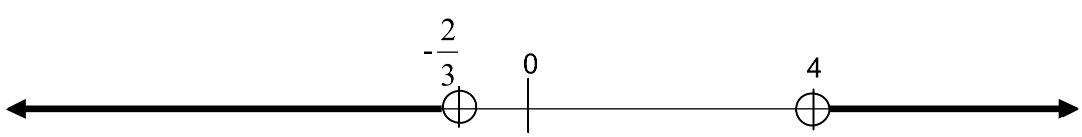
$\vert u \vert > c$	means that	$u > c$	or	$u < -c$
$\vert u \vert < c$	means that	$u < c$	and	$u > -c$
	means more concisely that $-c < u < c$			
provided that $c > 0$				

There is one very important thing to note in this general formula: the difference between the word “and” versus the word “or.” Look back to the previous examples. Given two inequalities, saying “or” means that <u>either</u> of the two inequalities can be true. Saying “and” means that <u>both</u> of the given inequalities must be true. Saying $x < 2$ <u>and</u> $x > -2$ means that x must be somewhere between -2 and 2 on the number line. Saying $x < 2$ <u>or</u> $x > -2$ means essentially that x could be any real number. Saying that $x > 2$ <u>or</u> $x < -2$ is a way of indicating x could be any real number outside of the interval from -2 to 2. Saying that $x > 2$ <u>and</u> $x < -2$ means that there is no solution. It seems like a list of facts to memorize, but in reality there is only one fact. “And” means that both conditions must be true while “or” means that only one is required to be true. We can finish up our work with absolute value inequalities with one last example.

Example:
Solve $| -3q + 5 | > 7$ for q.

Solution:
Given the form of the question, we know that we break apart the given expression into two inequalities joined by an “or.”

$-3q + 5 > 7$	or	$-3q + 5 < -7$	← break the absolute value inequality apart
$-3q > 2$	or	$-3q < -12$	← add the additive inverse of 5 to each side
$q < -\frac{2}{3}$	or	$q > 4$	← flip the inequality signs



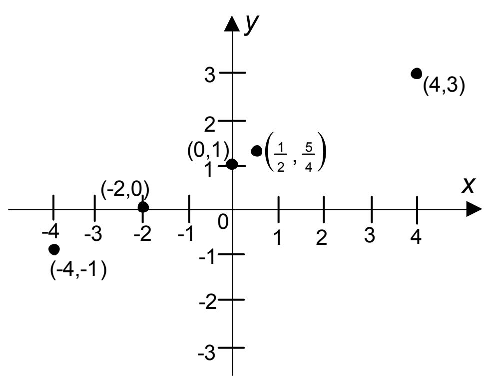
WHEN TWO VARIABLES LOVE EACH OTHER VERY MUCH…

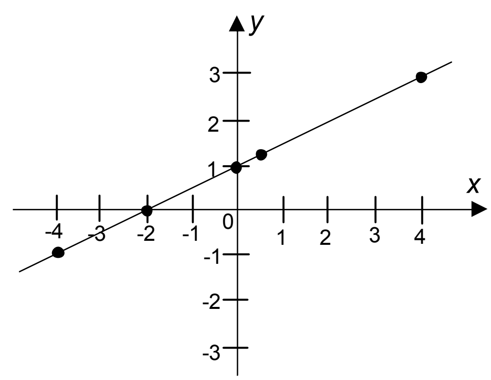
Ordered Pairs **Point-Slope Formula** **Slope-Intercept Form** **Standard Form**
Abscissa **Ordinate** **Cartesian-Coordinate** **x-intercept**
Slope **Origin**

Thus far, we have discussed equations of one variable. We have worked with equations in which we solved explicitly for the possible value(s) of x, y, z, n, m, t, or whatever variable is named. What happens, then, if an equation has more than one variable? What if we are dealing with an equation like “$-2x + 6y - 4 = x + 2$” ? We now have two variables, x and y. If we try solving the equation for one of the variables, we’ll get an expression containing the other variable instead of a number.

Solving for x gives us the equation x = 2y - 2. Similarly, solving for y gives the equation y = $\frac{1}{2}$x + 1. Clearly, we need some new ideas. What sorts of numbers can satisfy the equation? Maybe we can rely on our old friend logic to find a few combinations of x and y that make the equation true. One such solution is "x = 4, y = 3," while another is "x = -2, y = 0," and still another is "x = 0, y = 1."

What we can choose to do is this: we can represent all of these combinations of x and y as **ordered pairs** of numbers. The three combinations of x and y above would be written as (4, 3), (-2, 0), and (0, 1)—in each case, we write the x value as the first of two numbers, hence the term "ordered pair." Other possible examples of ordered pairs that satisfy this equation are $\left(\frac{1}{2}, \frac{5}{4}\right)$ and (-4, -1). If, as mathematicians,[11] we want a way to organize all of the possible solutions to this linear equation at the same time, we can graph these ordered pairs on a two-dimensional plane with the first number, or the **abscissa**, representing the x-coordinate and the second number, the **ordinate**, representing the y-coordinate. This two-variable equation has an infinite number of solutions; the five ordered pairs listed above appear below left.





In the left graph, we see the five points on the coordinate-plane. The idea of using two numbers to represent a place on a plane is known as the **Cartesian-Coordinate** system. The primary thing that we notice about the graph on the left is that the five points that are all solutions to the equation appear to be lying on a straight line. On the right, we confirm our guess and show that the five points are indeed on a straight line. *Any linear equation (an equation with no exponents) that has two variables "x" and "y" has an infinite number of solutions, and those solutions can be graphed onto a plane as a straight line that extends infinitely in both directions.* The graphs as pictured here do not extend forever, but in actuality, even the point (200, 101) exists on the line and is a solution to the equation.

The equation itself, "-2x + 6y – 4 = x + 2," gives us much information. Using a bit of algebraic rearranging, we can transform this to an equivalent equation, 3x – 6y = -6. An equation with two variables in this *ax + by = c* form is said to be in **standard form**.

Example:
Rewrite the two-variable equation 12x – 3y = 9 + 17x – y + 2 in standard form.

Solution:
12x – 3y = 9 + 17x – y + 2
12x – 3y = 17x – y + 11 ← Commutative Property
-5x – 3y = - y + 11 ← Add the additive inverse of 17x to both sides
-5x – 2y = 11 ← Add the additive inverse of -y to both sides
5x + 2y = -11 ← Multiply both sides by -1 so that the first number is positive
The last step here is entirely optional. It seems to be mathematical custom to make the x-coefficient *a* positive in *ax + by = c*, but either of the last two lines could be considered the standard form of the equation.

[11] If you're not a mathematician, then at least pretend you are for the time being.

What else can we say about the graphs above? There is a point, called the **x-intercept**, where the line intersects the x-axis. That x-intercept is (-2,0). There is another point, called the **y-intercept**, where the line intersects the y-axis. That y-intercept is (0,1).

Example:
What is the only point that can be both an x-intercept and a y-intercept for the same line?

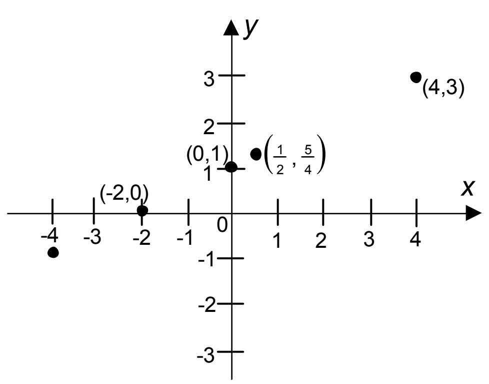
Solution:
For a point to be a line's x-intercept and y-intercept simultaneously, it must be on both axes. The only such point is the point (0,0), known as the **origin**.

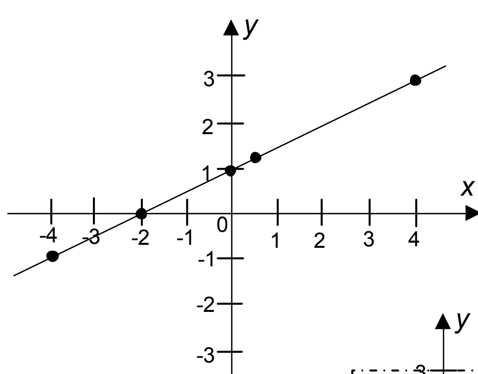
All graphed lines with have both an x-intercept and a y-intercept, with the exception of completely horizontal and completely vertical lines.

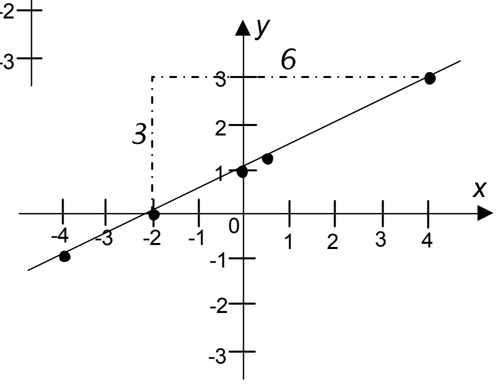
Example:
What are the x-intercept and y-intercept of the standard form line $3x + 7y = 84$?

Solution:
The x-intercept of a line occurs when y = 0. Thus, we can find the x-intercept by substituting y = 0 into the equation.
$3x + 7(0) = 84$
$3x = 84$
$x = 28$
The x-intercept is (28,0).
The y-intercept of the line then will occur when x = 0. The y-intercept can then be found when we substitute x = 0 into the equation.
$3(0) + 7y = 84$
$7y = 84$
$y = 12$
The y-intercept is (0,12).

There is one other descriptor of lines: their steepness, or **slope**. In algebra classes, a line's slope is commonly taught as "rise over run." What that means mathematically is that to find the slope of a line, you take the vertical change and divide by the horizontal change between any two arbitrary points on the line. For example, if we revisit the line we graphed earlier, we have five points already labeled on the line. (Remember that the line has an infinite number of points on it – we happen to have five conveniently labeled.) If we take any two of these points and calculate the vertical change divided by the horizontal change (rise divided by run), we can find the slope.







Example:
Find the slope of the line above.

Solution:
We want to find vertical change over horizontal change. This means we want to find change in "y" and divide by change in "x." I arbitrarily pick two points: in this case, I'll choose (-2,0) and (4,3). y goes from 0 to 3 so the change in y is 3. x goes from -2 to 4 so the change in x is 6. The

slope is then $\frac{3}{6}$, or $\frac{1}{2}$. Note that we could have taken the points in the reverse order; the final answer would have been the same. If we had said that y goes from 3 to 0, the change in y would have been -3. x going from 4 to -2 would have given a change of -6. That means there would have been a slope of $\frac{-3}{-6}$, or $\frac{1}{2}$.

Frequently, rather than expressing equations in standard form ($ax + by = c$), mathematicians prefer expressing equations in **slope-intercept form**, or $y = mx + b$ form.

Example:
Express the equation $2x - 4y = -12$ in slope-intercept form, and find the line's x-intercept and y-intercept.

Solution:
$2x - 4y = -12$
$-4y = -2x - 12$ ← add the additive inverse of 2x to each side
$y = -\frac{1}{4}(-2x - 12)$ ← multiply by the multiplicative inverse of -4 on each side
$y = \frac{1}{2}x + 3$ ← distributive property
To solve for the x-intercept, we substitute $y = 0$:
$0 = \frac{1}{2}x + 3$
$-3 = \frac{1}{2}x$ ← add the additive inverse of 3 to each side
$-6 = x$ ← multiply by the multiplicative inverse of $\frac{1}{2}$ on each side
The x-intercept is (-6,0).
To solve for the y-intercept, we substitute in $x = 0$:
$y = \frac{1}{2}(0) + 3$
$y = 3$
The y-intercept is (0,3).

Because the substitution of $x = 0$ allows us to find the y-intercept, we know that in slope-intercept form $y = mx + b$, (0,b) must be the y-intercept. In the example problem above, (0,3) was the y-intercept. This allows us to graph lines very quickly if they are given in slope intercept form. "m" is the slope, and "b" is the y-intercept.

Example:
Find the slope and y-intercept of
a) $13x + 12y = -5$
b) $mx + ny = p$

Solution:
a) $13x + 12y = -5$
$12y = -13x - 5$ ← add the additive inverse of 13x to each side
$y = -\frac{13}{12}x - \frac{5}{12}$ ← multiply by the multiplicative inverse of 12 on each side
The slope is $-\frac{13}{12}$, and the y-intercept is $-\frac{5}{12}$.
b) $mx + ny = p$
$ny = -mx + p$ ← add the additive inverse of mx to each side
$y = -\frac{m}{n}x + \frac{p}{n}$ ← multiply by the multiplicative inverse of n on each side
The slope is $-\frac{m}{n}$, and the y-intercept is $\frac{p}{n}$.

Example:
By rearranging into slope-intercept form, quickly graph the lines
a) $12x + 15y = 30$
b) $3x - 4y = -12$
c) $y - \frac{1}{2} = 0$

Solution:

a) $12x + 15y = 30$

$15y = -12x + 30$ ← add the additive inverse of 12x to each side

$y = -\frac{4}{5}x + 2$ ← multiply by the multiplicative inverse of 15 on each side

We know that this line must intersect the y-axis at (0,2) and have a slope of $-\frac{4}{5}$. In the graph below for (a), there is a rise of -4 (a fall of 4) proportional to a run of 5.

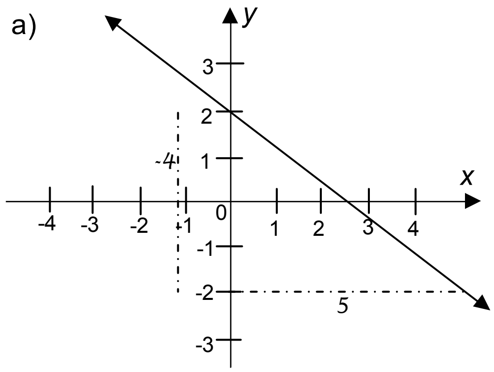
b) $3x – 4y = -12$

$-4y = -3x – 12$ ← add the additive inverse of 3x to each side

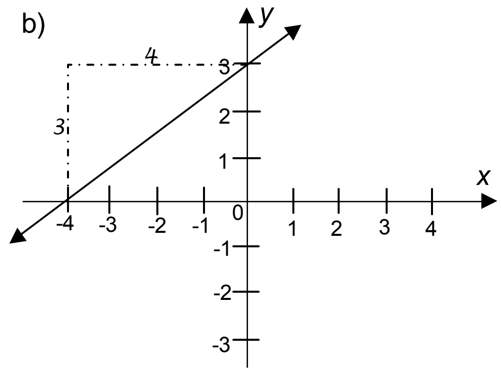
$y = \frac{3}{4}x + 3$ ← multiply by the multiplicative inverse of -4 on each side

This line must now have a y-intercept of (0,3) and a slope of $\frac{3}{4}$. In the graph for (b), there is a rise of 3 proportional to a run of 4.

a)



b)

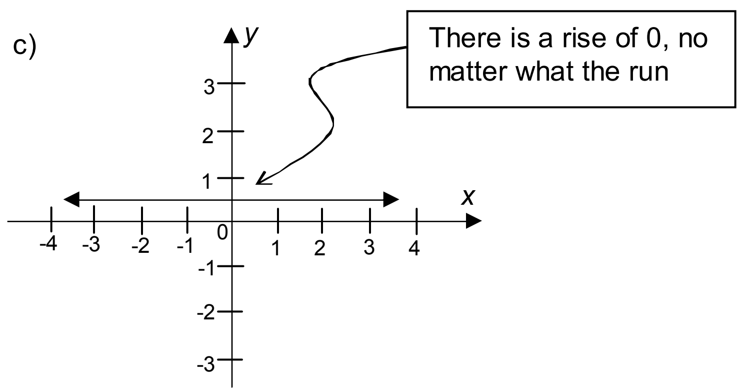


c) $y – \frac{1}{2} = 0$

$y = \frac{1}{2}$ ← add the additive inverse of $-\frac{1}{2}$ to each side

$y = 0x + \frac{1}{2}$ ← add the additive identity element to the right expression

c)



The last example was written to set the stage for another lesson concerning lines. Equations of the form $y = c$ or $x = c$ create horizontal and vertical lines, respectively. People often forget which type of equation creates which line. Remember, though, that the line resulting from an equation is a graph of all the points that can satisfy the equation. With that in mind, the graph of $x = 2$ must contain the points (2,0), (2,-3), (2,5), (2,-10), (2,7), etc. If those points are graphed on a Cartesian Coordinate plane, then they will form a vertical line. Likewise, a graph of the equation $y = -3$ contains all of the points (0,-3), (5,-3), (-2,-3), (12,-3), etc. and forms a horizontal line. In addition, since slope is defined as $\frac{rise}{run}$, a horizontal line has a slope of 0 (no rise with arbitrary run) while a vertical line has an undefined slope (arbitrary rise divided by zero run).[12]

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[12] Remember that any division by 0 is always undefined. In fact, division by 0 is one of the seven cardinal no-nos of mathematics. I'll make up the other six later.

Example:
What are the equations of the x and y axes?

Solution:
The x-axis is a horizontal line that crosses the y-axis at the point (0,0). The x-axis must have equation $y = 0$. The y-axis is a vertical line that crosses the x-axis at the point (0,0). The y-axis must then have $x = 0$ as its equation.

Example:
What is the equation in slope-intercept form of a line that passes through (-2,3) and (3,5)?

Solution:
The first logical thing to do in this case is find the slope. We are already given two points on the line, so all we must calculate is the change in *y* and the change in *x*. The slope must then be $\frac{2}{5}$, and we know that $m = \frac{2}{5}$ in the equation $y = mx + b$. We now need a logical way to find *b* in the equation. This equation must be true for all of the points along the line, including the two we were already given; intuitively, if we substitute one of the given points into the equation, we can solve for the missing variable *b*. I'll arbitrarily choose the second point (3,5) and substitute.

$y = mx + b$
$5 = \frac{2}{5}(3) + b$ ← substitution of what we know (the slope and one point)
$\frac{19}{5} = b$ ← add the additive inverse of $\frac{6}{5}$ to each side

We already knew the slope and have now solved for the y-intercept. Thus, the equation in slope-intercept form is
$y = \frac{2}{5}x + \frac{19}{5}$

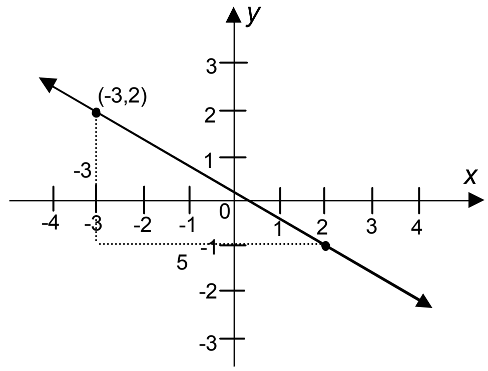
The above example illustrates one way of finding the equation of a line given two points (or one point and the slope). Substitution into the slope-intercept form is one very intuitive method of finding the equation of a line. Another method is the substitution into the **point-slope formula**. Given slope *m* and a point (x_1,y_1) on a line, we can solve for the equation of the line using the formula $y - y_1 = m(x - x_1)$. We can also use the formula in reverse to quickly graph a line given its point-slope form.

Example:
Use the point-slope formula to find the equation of a line with slope $\frac{2}{5}$, passing through the point (-2,3).

Solution:
We recognize this as the same line that was found above. We should get the same answer.
$y - y_1 = m(x - x_1)$ ← point-slope formula
$y - 3 = \frac{2}{5}(x - (-2))$ ← substitution of what we know
$y - 3 = \frac{2}{5}x + \frac{4}{5}$ ← distributive property
$y = \frac{2}{5}x + \frac{19}{5}$ ← add the additive inverse of -3 to each side

Example:
Quickly graph the equation $y - 2 = -\frac{3}{5}(x + 3)$

Solution:
At first, we may be tempted to rearrange this equation into slope-intercept form, but in the point-slope formula, it is already ripe for graphing. We see that a point on the line is (-3, 2); we also know there is a slope of $-\frac{3}{5}$. Those two facts alone are enough to form a graph.

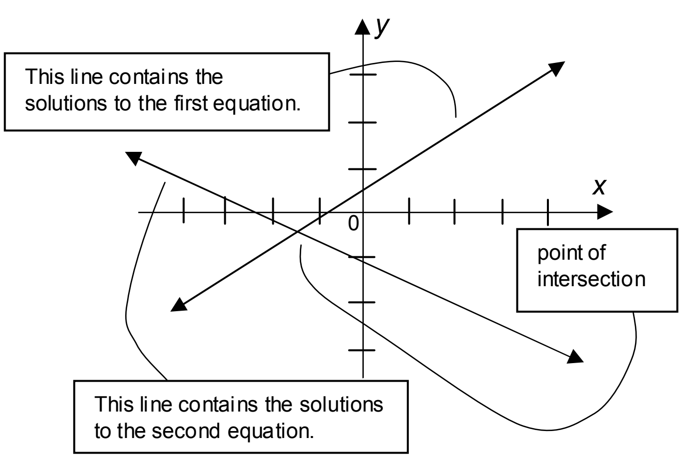


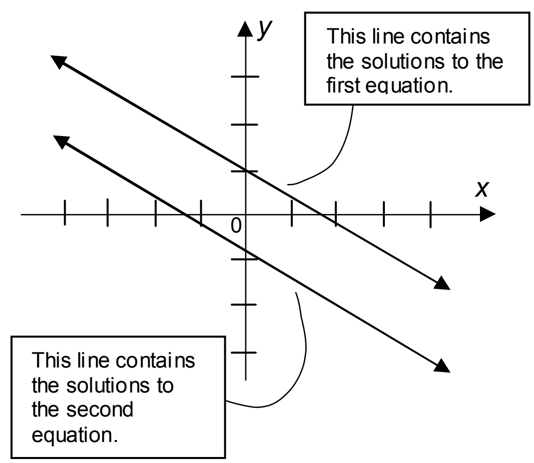
Remember, there are three different forms for the equation of a line: standard form (ax + by = c), point-slope form ($y - y_1 = m(x - x_1)$), and slope-intercept form (y = mx + b). Each form has different properties with which you should be familiar, and which form is most appropriate will have to be determined on a case-by-case basis.

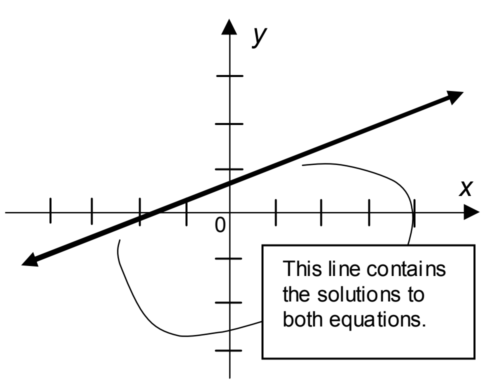
SYSTEMS OF EQUATIONS

Independent **Inconsistent** **Dependent**

We have just finished examining that linear two-variable equations have an infinite number of solutions; those solutions can be "graphed" to form a straight line. What happens, then, if we have *two* linear equations, each containing the same two variables? Is there exactly one solution that satisfies both equations? Frequently, the answer is yes.







In the upper leftmost graph, two lines intersect at one point. If one line contains all of the solutions to one equation and the other line contains all of the solutions to the other equation, the intersection is the one and only solution to both equations. Two equations such as these are known as **independent** equations. In the upper right graph, the two lines are parallel and do not intersect. In this case, there is no solution which satisfies both equations simultaneously; such equations are said to be **inconsistent**. Lastly, in the lower graph, two lines coincide. This can only occur if the two equations are actually equivalent; all of the points along the line(s) then satisfy both equations, and the equations are termed **dependent**. It is a major rule of algebra that to solve several equations simultaneously, one must have at least as many independent equations as one has unknowns.[13]

To solve these "systems" of two equations, there are several methods we could choose to use. As the pictures above illustrate, we could choose to graph the solutions of the two equations and see what point(s), if any, satisfy both equations. The method of graphing to solve systems has two drawbacks though: it is slow, and unless the slopes and intercepts are "nice" numbers, it is inaccurate and subject to visual error. We need other methods. The first major method of solving simultaneous equations is the method of *substitution*. For substitution, solve for one variable using one equation, then substitute that expression into the second equation. Another example is in order.

Example:
Solve the system of equations below by substitution.
$x - 4y = -13$
$5x + 2y = 1$

Solution:
In the first equation, x has no coefficient and can be easily isolated.

$x - 4y = -13$	← [first equation]
$x = 4y - 13$	← add the additive inverse of $-4y$ to each side
$5x + 2y = 1$	← [second equation]
$5(4y - 13) + 2y = 1$	← substitute $4y - 13$ in place of x (as it was solved for above)
$20y - 65 + 2y = 1$	← distributive property
$22y - 65 = 1$	← commutative property
$22y = 66$	← add the additive inverse of -65 to each side
$y = 3$	← multiply both sides by the multiplicative inverse of 22
$x = 4y - 13$	← solved for above; restatement of line 2
$x = 4(3) - 13$	← substitute $y = 3$
$x = -1$	← simplification

The solution of the equation is $x = -1$, $y = 3$, or (-1,3).

The second major method of solving systems of equations is known as *elimination*, also called *linear combination* in many textbooks. To solve a system of equations by elimination, we form equivalent equations that can be added together in a useful way. In other words, we transform the equations such that one variable "cancels out." This explanation makes more sense in an application than it does in a paragraph form; yet another example is in order.

Example:
Solve the system of equations below by elimination.
$3x + 12y = 19$
$6x - 9y = 5$

[13] This algebraic rule comes in very handy in physics, where solving simultaneous equations actually has a practical purpose. This is an answer to those asking, "When will I need this in life?" So there.

Solution:
Look at the x-coefficients. If we multiply the first equation by -2, the x terms will become additive inverses.

$3x + 12y = 19$	← [first equation]
$-6x - 24y = -38$	← multiply both sides by -2 to create an equivalent equation
$\underline{6x - 9y \quad = \quad 5}$	← [second equation]
$-33y = -33$	← add the two equations, term by term, to create a new equation
$y = 1$	← multiply both sides by the multiplicative inverse of -33
$3x + 12(1) = 19$	← substitute y = 1 into one of the original equations
$3x = 7$	← add the additive inverse of 12 to each side
$x = \frac{7}{3}$	← multiply both sides by the multiplicative inverse of 3

The solution is $(\frac{7}{3}, 1)$.

In the above example, there was a simple multiplication that resulted in a cancellation of variables. The next problem presents a more complicated example; nevertheless, the concept remains the same. Also, many people prefer to work the elimination method in a combination of horizontal and vertical calculations. The explanations have been left out of each step, showing the work in a logical vertical and horizontal manner.

Example:

Solve the system $\begin{matrix} 3x + 11y = 17 \\ 7x - 7y = 13 \end{matrix}$ by elimination.

Solution:

$$3x + 11y = 17 \xrightarrow{\times 7} 21x + 77y = 119$$

$$7x - 7y = 13 \xrightarrow{\times(-3)} -21x + 21y = -39$$

$$98y = 80$$

$$3x + 11\left(\frac{40}{49}\right) = 17 \longleftarrow y = \frac{40}{49}$$

$$3x + \frac{440}{49} = \frac{833}{49}$$

$$3x = \frac{393}{49}$$

$$x = \frac{131}{49}$$

The solution to the system is $\left(\frac{131}{49}, \frac{40}{49}\right)$.

In this example, the system of equations presents a little more difficulty. The x terms do not cancel with a simple multiplication as in the earlier example. Instead, the x terms must have a common multiple found so that they can cancel out. In this case, that multiple is 21, and this provides us our “jumping off point” for the elimination. In the two worked examples above, we chose to cancel x and find y first; remember that this is only an arbitrary choice – y could have been cancelled first if we had wanted.

Systems of independent equations can be solved by either substitution or elimination, but what should be done with systems of dependent or inconsistent equations? Recall that a system of two dependent equations will coincide so that there are an infinite number of solutions; a system of two inconsistent equations will run parallel so that there are no possible solutions to the system. Two dependent equations can be transformed to equivalent equations; inconsistent equations can be recognized because, as parallel lines, they will have identical slopes but different y-intercepts.

Examples:
Characterize the following two systems are either inconsistent or dependent.

(1) $2x - 6y = 12$
$x - 3y = 8$

(2) $14x + 7y = 21$
$-21x - 10.5y = -31.5$

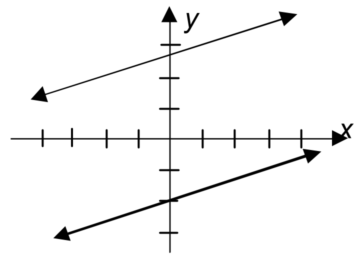
Solutions:

(1) $2x - 6y = 12$ $\quad x - 3y = 8$

$-6y = -2x + 12$ $\quad -3y = -x + 8$

$y = \frac{1}{3}x - 2$ $\quad y = \frac{1}{3}x + \frac{8}{3}$

These two equations have identical slope but differing y-intercepts. They form an inconsistent system and there is no common solution.

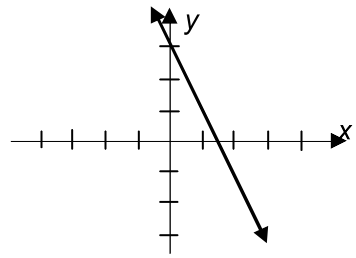


(2) $14x + 7y = 21$ $\quad -21x - 10.5y = -31.5$

$7y = -14x + 21$ $\quad -10.5y = 21x - 31.5$

$y = -2x + 3$ $\quad y = -2x + 3$

These two equations are actually equivalent. This means the graphs will coincide and have infinite common solutions. This creates a dependent system.



If, in an overzealous spout of alge-mania[14], you attempted to solve a system of (presumably independent) equations, and substitution or elimination yielded a statement such as "-3 = 0" or "5 = 5," you would know that there are either no solutions or an infinite number of solutions, respectively. Remember, an algebraic truth means that there are tons of possible solutions, and an algebraic untruth means that there are none.

As an additional aside, you should also know that two lines that are perpendicular have slopes that are additive multiplicative inverses of each other (we more commonly say that one slope is the negative reciprocal of the other). That is, for two lines to be graphed perpendicularly to each other, one will have slope m and the other will have slope $-\frac{1}{m}$. This also means that if we have two slopes such that $slope_1 \times slope_2 = -1$, the respective lines are perpendicular.

BIGGER SYSTEMS OF EQUATIONS

Real headaches begin when more than two variables are involved in a system. With only two variables, we can easily graph and/or visualize coincident, parallel, and intersecting lines to understand the concepts of systems that have different numbers of solutions. If we have three or more variables, however, the system must be visualized in three (or more? Yikes!) dimensions. We won't let that stop us. As long as there are as many independent equations as there are variables, we can still find one distinct solution to the system.

The standard methods used for two-variable systems, substitution and elimination, are also used to solve three-variable systems. In the case of these larger systems though, the methods will often have to be used repeatedly in order to make any headway in solving the system.

Example:

Solve the system below first by substitution and then by elimination.

$2x + y - 4z = -19$

$-4x + 2y + 3z = 8$

$12x - 6y + 3z = 0$

[14] Not to be confused with DemiDec's board game, AcaMania.

Solution by Substitution:
If we solve for y in the first equation then substitute into the second and third equations, we will create a two-variable system of two equations.

$2x + y - 4z = -19$	
$y = 4z - 2x - 19$	← solve for y in the first equation
$-4x + 2(4z - 2x - 19) + 3z = 8$	← substitute y into the second equation
$-4x + 8z - 4x - 38 + 3z = 8$	← distributive property
$-8x + 11z = 46$	← simplify
$12x - 6(4z - 2x - 19) + 3z = 0$	← substitute y into the third equation
$12x - 24z + 12x + 114 + 3z = 0$	← distributive property
$24x - 21z = -114$	← simplify
$24x = 21z - 114$	← add the additive inverse of -21x to each side
$x = \frac{21z-114}{24}$	← multiply by the multiplicative inverse of 24
$x = \frac{7z-38}{8}$	← simplify the fraction
$-8\left(\frac{7z-38}{8}\right) + 11z = 46$	← substitute into the 5th line
$-7z + 38 + 11z = 46$	← distributive property
$4z = 8$	← add the additive inverse of 38 to each side
→ $z = 2$	← multiply by the multiplicative inverse of 4
$2x + y - 4z = -19$	← rewrite the first equation
$2x + y - 4(2) = -19$	← substitute the value of z into the first equation
$2x + y = -11$	← add the additive inverse of -8 to each side
$y = -2x - 11$	← solve for y now
$-4x + 2y + 3z = 8$	← rewrite the second equation
$-4x + 2(-2x - 11) + 3(2) = 8$	← substitute both y and z into this equation
$-4x - 4x - 22 + 6 = 8$	← distributive property
$-8x = 24$	← add the additive inverse of -16 to each side
→ $x = -3$	← multiply by the multiplicative inverse of -8
$y = -2x - 11$	← rewrite an equation from six lines up
$y = -2(-3) - 11$	← substitute x into this equation
→ $y = -5$	← simplify

The solution to the system is thus (-3, -5, 2). As you can see, substitution is quite tedious when applied to a system of three variables. Often, a combination of substitution and elimination, or even the exclusive use of elimination, is easier.

Solution by Elimination:

$2x + y - 4z = -19$
$-4x + 2y + 3z = 8$
$12x - 6y + 3z = 0$

This system is very rare in that it is peculiarly easy. Most often, an elimination will eliminate only one variable. Two eliminations will create a two-variable system with two variables. In this case, however, not just one, but *two* variables are eliminated from the 2nd and 3rd equations at the same time.

×3

$-12x + 6y + 9z = 24$	← multiply the second equation by 3
$12x - 6y + 3z = 0$	← rewrite the third equation
$12z = 24$	← add these lines
→ $z = 2$	← multiply by the multiplicative inverse of 12
$2x + y - 4(2) = -19$	← substitute z into the first equation
$2x + y = -11$	← add the additive inverse of -8 to each side
×2 $-4x + 2y + 3(2) = 8$	← substitute z into the second equation
$-4x + 2y = 2$	← add the additive inverse of 6 to each side
$4x + 2y = -22$	← multiply that above equation by 2
$4y = -20$	← add the two lines above
→ $y = -5$	← multiply by the multiplicative inverse of 4
$2x + (-5) = -11$	← substitute y into an above equation
$2x = -6$	← add the additive inverse of -5 to each side
→ $x = -3$	← multiply by the multiplicative inverse of 2

The solution to the system is thus (-3, -5, 2), and we have found it with significantly less work than the method of substitution.

Most commonly, a three-variable system will lend itself to a combination of both substitution and elimination. Another example is shown below with the explanations removed and logical arrows added.

Example:
Solve the system below.

$2x - 3y + 5z = 7$
$-3x + 2z = -15$
$9x + y - 4z = 46$

Solution:

$2x - 3y + 5z = 7$ —(1)→ $y = \dfrac{2x + 5z - 7}{3}$

$-3x + 2z = -15$

$9x + 6y - 4z = 51$ —(2)→ $9x + 6\left(\frac{2x+5z-7}{3}\right) - 4z = 51$

$9x + 4x + 10z - 14 - 4z = 51$

$13x + 6z = 65$

(3) ×(-3): $9x - 6z = 45$

(4) $22x = 110$

$x = 5$

(5) $9(5) - 6z = 45$

$-6z = 0$

$z = 0$

(6) $y = \dfrac{2x + 5z - 7}{3}$

$y = \dfrac{2(5) + 5(0) - 7}{3}$

$y = 1$

The solution is (5,1,0). Note that many substitutions were used in this case, and only one elimination was convenient. The work is a bit cluttered and hard to examine, but the first step is the solving for y in the first equation. The equation is transformed from $2x - 3y + 5z = 7$ to

$y = \frac{2x + 5z - 7}{3}$. Step 2 is the substitution of y into the third equation, which then simplifies to 13x + 6z = 65. Steps 3 and 4 involve transforming the original 2^{nd} equation and the subsequent elimination of z from the new two-variable system containing x and z. Step 5 is then the substitution of x into the modified 2^{nd} equation in order to find z, and step 6 is the substitution of x and z to find y. There is a lot going on in many directions at once. Pause briefly to soak up the massive mathematical manipulation manifest in this model and multi-dimensional miracle.[15] Also, I should make mention of the matrices made available on many models of modern calculators.[16]

The mathematics of matrices are beyond the scope of this resource, but you should feel free to consult a solid second-year algebra textbook to understand the inner workings of matrices; they provide a direct way of solving systems.

Example:
Solve the system below.
2x – 3y + 5z = 7
-3x + 2z = -15
9x + y – 4z = 46

Solution:

$$\begin{bmatrix} 2 & -3 & 5 \\ -3 & 0 & 2 \\ 9 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -15 \\ 46 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 0 & 2 \\ 9 & 1 & -4 \end{bmatrix}^{-1}\begin{bmatrix} 7 \\ -15 \\ 46 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/37 & 7/37 & 6/37 \\ -6/37 & 53/37 & 19/37 \\ 3/37 & 29/37 & 9/37 \end{bmatrix}\begin{bmatrix} 7 \\ -15 \\ 46 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

The solution as an ordered triple is (5,1,0).

THE POLYNOMIALS' FRIEND, THE RATIONAL EXPRESSION

GCF **FOIL** **Rational Expression**

It is now time to reexamine the ideas and concepts of polynomials. Earlier, a polynomial was defined most simply as any sum of monomials; for all the mathematicians out there, a polynomial was defined as "Any expression that can be written in the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$, where each a_i is a constant and n is an integer." For a brief, (seemingly) pointless digression, let's work some binomial multiplication together. (A binomial is a polynomial with exactly two terms.)

Examples:
Multiply the following:
a) (x – 1)(x + 2)
b) (x + 3)(x + 5)
c) (x – 4)(3x + 2)

[15] …unable to abstain from the alliterative allure.
[16] Good grief! I've never seen so many "m" words in my life!

d) (a + b)(c + d)
e) (x – 7)(x + 7)

Solutions:
In multiplying binomials, many of us learned the **FOIL** acronym in school: we multiply together the first terms, the outside terms, the inside terms, and the last terms, then combine like terms. The terms do not have to be multiplied in that order, but it is usually the most logical course of action.

a) Multiply first terms: x and x, inside terms: -1 and x, outside terms: 2 and x, and last terms: -1 and 2. The like terms 2x and -1x can combine for +x.
$(x - 1)(x + 2) =$
$x^2 + 2x - 1x - 2 =$
$x^2 + x - 2$
b) Following the same procedure as above, we have
$(x + 3)(x + 5) =$
$x^2 + 5x + 3x + 15 =$
$x^2 + 8x + 15$
c) $(x - 4)(3x + 2) =$
$3x^2 + 2x - 12x - 8 =$
$3x^2 - 10x - 8$
d) $(a + b)(c + d) =$
$ac + ad + bc + bd$ ← no like terms to combine
e) $(x - 7)(x + 7) =$
$x^2 + 7x - 7x - 49 =$
$x^2 - 49$

Example (d) is the general formula for binomial multiplication, and example (e) is a special case in which the two "x terms" add to 0 and we are left with only the difference of two squared terms.

Now, what if we were asked to *reverse* a binomial multiplication? Many decathletes will recognize such a request as a prompt to begin *factoring*, in which one expression is rewritten as a product of other expressions.

Examples:
Factor the following expressions:
a) $x^2 - 7x + 12$
b) $x^2 + 2x - 15$
c) $2x^2 - 3x - 5$
d) $10x^3 - 15x^2 - 25x$
e) $3x^2 - 75$

Solutions:
a) By observing the examples above, we realize that our factorization will likely be in the form (x + a)(x + b). a and b must multiply to give 12, and they must add to -7. The only possible solution, then, is that a = -3 and b = -4. (The order is not important, however; I could have also said that a = -4 and b = -3.) Thus, the factorization is
$x^2 - 7x + 12 =$
$(x - 3)(x - 4)$
b) This plan of attack is the same as above. In this case, a and b must multiply to give -15, and they must add to 2. a = -3 and b = 5 now satisfy the conditions.
$x^2 + 2x - 15 =$
$(x - 3)(x + 5)$
c) This trinomial[17] is a bit harder to factor than the first two, primarily because of that leading coefficient. We know that "first × first" equals $2x^2$ so the factorization must now take the form (2x + a)(x + b), and now, a and b must multiply to -5, but 2b + a will add

[17] In case the prefix does not make it obvious to you, a trinomial is a polynomial consisting of exactly three terms.

to -3. This gives us the solution for our factorization that $a = -5$ and $b = 1$. Thus,
$2x^2 - 3x - 5 =$
$(2x - 5)(x + 1)$

d) Before launching into a homicidal rage upon seeing a problem like this, we need only to examine what we know and what we must do. This first term contains x^3, a problem we have not encountered before. In a flash of genius, however, we notice that each term is a multiple of the same factor: 5x. Let's first factor that term out by itself, sort of a reverse of the distributive property, if you will.
$10x^3 - 15x^2 - 25x =$
$5x(2x^2 - 3x - 5) =$ ← factor out the 5x. Ha! We've seen this trinomial.
$5x(2x - 5)(x + 1)$

e) As we did in example (d) above, we'll factor out a common term: this time, it's 3. After that, we are left with the expression $x^2 - 25$, one of those "special forms" with no middle term. Because of the previous examples, this should be no problem.
$3x^2 - 75 =$
$3(x^2 - 25) =$
$3(x - 5)(x + 5)$

Although these example illustrate how the factoring process works in general, the only way to become a really competent factorer is to practice factoring. There are a few aids, though, that should prove helpful in factoring polynomials: (1) Always look first for a **GCF**—the greatest common factor shared by the polynomial's terms, and (2) Memorize the factoring patterns for a difference of squares, a difference of cubes, and a sum of cubes, and (3) Learn that a sum of squares *cannot* be factored with real numbers.

A Difference of Two Squares
$x^2 - y^2 = (x - y)(x + y)$

A Difference of Two Cubes	A Sum of Two Cubes
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Examples:
In each of the following three expressions, first state the GCF. Then factor each expression as completely as possible.

a) $5nxm^2 - 125xn^3$
b) $4af^2x^3 + 108a^4f^2y^3$
c) $1458p^6 - 128q^6$

Solutions:

a) In this case, the GCF is 5nx.
$5nxm^2 - 125xn^3 =$
$5nx(m^2 - 25n^2) =$ ← factor out the GCF
$5nx(m - 5n)(m + 5n)$ ← follow the pattern for a difference of two squares

b) The GCF here is $4af^2$.
$4af^2x^3 + 108a^4f^2y^3 =$
$4af^2(x^3 + 27a^3y^3) =$ ← factor out the GCF
$4af^2[x^3 + (3ay)^3] =$ ← rewrite the sum of two cubes to be more readable
$4af^2(x + 3ay)(x^2 - 3axy + 9a^2y^2)$ ← follow the pattern for a sum of two cubes

c) The GCF here is not difficult to find. It is simply 2. The two terms share nothing else in common.
$1458p^6 - 128q^6 =$
$2(729p^6 - 64q^6) =$ ← factor out the GCF
$2[(27p^3)^2 - (8q^3)^2] =$ ← rewrite the difference of two squares[18]
$2(27p^3 - 8q^3)(27p^3 + 8q^3) =$ ← factor the difference of two squares
$2[(3p)^3 - (2q)^3][(3p)^3 + (2q)^3] =$ ← rewrite the cube patterns to be more readable
$2(3p - 2q)(9p^2 + 6pq + 4q^2)(3p + 2q)(9p^2 - 6pq + 4q^2)$

Now that we can factor these simple polynomials (perhaps example (c) wasn't so simple) at a satisfactory level, we must turn our attention to a dear friend of the polynomial: the **rational expression**. A rational expression is any expression that can be written as a ratio of polynomials.

Example:
List some rational expressions.

Solution:
$$\frac{x^2 + 5x - 6}{x^2 - 1}, \quad \frac{2x^4 - 19x^3 - 27x^2 + 279x + 405}{2x^4 + 13x^3 - 43x^2 - 297x - 315}, \quad x^3 - 4x^2 + 3x - 1, \quad 1$$

Note in particular the last two rational expressions out of the four listed. Remember that any constant can be a polynomial, even "1" alone. Thus, any polynomial is itself a rational expression because any polynomial can be written $\frac{\text{polynomial}}{1}$. 1 is then a rational expression because $\frac{1}{1}$ is a ratio of polynomials.[19] Rational expressions, in a manner of speaking, resemble fractions. Instead of having integers in the numerator and denominator, rational expressions have polynomials in those positions instead. It's time to revisit our examples for the arithmetic of fractions and the cancellation law. Recall the cancellation law from much earlier – it is rewritten here for your convenience. The cancellation law says, in essence, that when a common factor appears in both the numerator and the denominator of an expression, it can be cancelled out provided the denominator not equal 0.

Cancellation Law
$\frac{ab}{ac} = \frac{b}{c}$ as long as $a \neq 0$ and $c \neq 0$

Example:

Use the cancellation law to simplify $\frac{x^2 + 5x - 6}{x^2 - 1}$.

Solution:
We should first factor both the numerator and denominator to see if there are common factors that can cancel out. We practiced factoring polynomials earlier, so these should not phase us in the least.

$$\frac{x^2 + 5x - 6}{x^2 - 1} =$$
$$\frac{(x + 6)(x - 1)}{(x - 1)(x + 1)} =$$

[18] Be sure you know why this was treated as a difference of two squares instead of a difference of two cubes. If you don't know, try working this example with a difference of two cubes from this point.
[19] Redundant people, those mathematicians… and they repeat themselves often.

$\frac{x+6}{x+1}$, as long as $x - 1 \neq 0$ and $x + 1 \neq 0$.

Therefore, $\frac{x^2+5x-6}{x^2-1} = \frac{x+6}{x+1}$ as long as $x \neq 1$ and $x \neq -1$.

The procedure illustrated above is completely analogous to the process involved in reducing a fraction to its lowest terms. Rational expressions can also be added, subtracted, multiplied, and divided exactly as fractions are. As is the case with fractions, multiplication is carried out "straight across," and division is defined simply as multiplication by the reciprocal. Also as is the case with fractions, addition and subtraction are a bit more complicated, simply because they require the use of a common denominator before the numerators can be combined. Approximately 25 pages ago, this algebra resource detailed the procedures involved in the arithmetic of fractions in a step by step manner. Those examples are listed again below, along with an analogous problem involving rational expressions.

$\frac{1}{2}+\frac{1}{6}=$	$\frac{1}{x-2}+\frac{1}{x^2-5x+6}=$
$\frac{1\cdot 3}{2\cdot 3}+\frac{1}{6}=$	$\frac{1\cdot(x-3)}{(x-2)(x-3)}+\frac{1}{(x-2)(x-3)}=$
$\frac{3}{6}+\frac{1}{6}=$	$\frac{x-3}{x^2-5x+6}+\frac{1}{x^2-5x+6}=$
$\frac{4}{6}=\frac{2\cdot 2}{3\cdot 2}=\frac{2}{3}$	$\frac{x-3+1}{x^2-5x+6}=\frac{x-2}{(x-2)(x-3)}=\frac{1}{x-3}$, as long as $x \neq 2$ and $x \neq 3$

The first line of these addition problems is the original problem. The second displays the transformation required to produce a common denominator. In the case of this rational expression problem, a common denominator is easily evident; sometimes, however, finding a common denominator will take a bit more digging. After like terms are combined, the resulting numerator and denominator are factored and the cancellation law is applied. Let's now revisit and expand on the second example.

$\frac{1}{3}+\frac{1}{2}\bullet\frac{5}{6}=$	$\frac{1}{x+3}+\frac{1}{x-2}\bullet\frac{x+5}{2(x+3)}=$
$\frac{1}{3}+\frac{1\cdot 5}{2\cdot 6}=$	$\frac{1}{x+3}+\frac{1\cdot(x+5)}{2(x-2)(x+3)}=$
$\frac{1}{3}+\frac{5}{12}=$	$\frac{1}{x+3}+\frac{x+5}{2x^2+2x-12}=$
$\frac{1\cdot 4}{3\cdot 4}+\frac{5}{12}=$	$\frac{1\cdot 2(x-2)}{(x+3)\cdot 2(x-2)}+\frac{x+5}{2x^2+2x-12}=$
$\frac{4}{12}+\frac{5}{12}=$	$\frac{2x-4}{2x^2+2x-12}+\frac{x+5}{2x^2+2x-12}=$
$\frac{9}{12}=\frac{3\cdot 3}{4\cdot 3}=\frac{3}{4}$	$\frac{3x+1}{2x^2+2x-12}$

Note that this rational expression problem does not simplify in the end; there are no common factors in the numerator and denominator of the final answer. Note also that the proper order of operations still applies: the multiplication of the two original rightmost expressions is still performed before the addition. Also, the common denominator for these rational expressions is a real monster to find; but we can make that work easier by examining the rational denominators' factorizations.

Let's look at a completely original stand-alone example now.

Example:
Simplify the expression below.

$$\frac{-x^2}{x^2-6x+5}-\frac{-2x^2+4x+5}{x-1}\div(x-5)$$

Solution:
According to the standard order of operations, we must perform the division at the right before we can do the subtraction. Our first order of business is to apply the definition of division.

$$\frac{-x^2}{x^2-6x+5}-\frac{-2x^2+4x+5}{x-1}\div(x-5)=$$

$$\frac{-x^2}{x^2-6x+5}-\frac{-2x^2+4x+5}{x-1}\cdot\frac{1}{x-5}=$$ ← definition of division

$$\frac{-x^2}{(x-1)(x-5)}-\frac{-2x^2+4x+5}{(x-1)(x-5)}=$$ ← perform the multiplication

$$\frac{-x^2-(-2x^2+4x+5)}{(x-1)(x-5)}=$$ ← combine the numerators since there is already a common denominator

$$\frac{-x^2+2x^2-4x-5}{(x-1)(x-5)}=$$ ← definition of subtraction

$$\frac{x^2-4x-5}{(x-1)(x-5)}=$$ ← simplify

$$\frac{(x-5)(x+1)}{(x-1)(x-5)}=$$ ← factor the numerator in an attempt to use the cancellation law

$\frac{x+1}{x-1}$ provided $x\neq 1$ and $x\neq 5$. ← cancellation law

Be careful when simplifying the rational expressions—be sure to get the details straight. Make sure you always "distribute" subtraction across the entire numerator to which it applies. Remember that, just like "regular fractions", rational expressions are multiplied "straight across" and addition / subtraction require a common denominator. If a common denominator is not readily available, then one must be created.

IRRATIONAL NUMBERS

Rational Number **Irrational Number** **radicand**

The very beginning of this resource detailed an introduction to the real numbers. Real numbers, we said, are the numbers that can be matched to a position on the number line. Of course zero, all positive numbers, and all negative numbers qualify as real numbers, but there exist even more categories and divisions to describe the real numbers. Any terminating or repeating decimal can be represented as a ratio of two numbers (a fraction) and is a **rational number**. All of the remaining decimals on the number line that neither terminate nor repeat are known as **irrational numbers**.

The majority of irrational numbers arise as the roots of numbers. For example, the square root of 2, written as $\sqrt{2}$, is approximately equal to 1.4142…., but the decimal neither terminates nor repeats. The square root of 6, written $\sqrt{6}$, is approximately 2.4495, but it too neither terminates nor repeats. Many cubic roots, designated $\sqrt[3]{x}$ or $x^{1/3}$, are irrational, as are most 4th and higher degree roots of numbers. What does it mean, then, when the decathlon curriculum lists the simplification of square roots among the amalgam of tasks that should be performable by decathletes? A square root expression is considered simplified if the number under the root symbol (the **radicand**) has no perfect square factors and there are no square roots appearing in any denominator.[20] The simplification of square roots is yet another topic that is most easily taught and explained through examples.

Square Root Simplification Rules
As long as $x \ge 0$ and $y \ge 0$, then $\sqrt{x^2} = x$, $\sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y}$, $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

Examples:
Simplify the following.

a) $\sqrt{75}$

b) $\sqrt{48}$

c) $\sqrt{700}$

d) $\sqrt{16} + \sqrt{20}$

e) $\sqrt{48} - \sqrt{75}$

f) $\sqrt{20} + \sqrt{125} - \sqrt{27}$

Solutions:

a) $\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$

b) $\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$

c) $\sqrt{700} = \sqrt{100 \cdot 7} = \sqrt{100} \cdot \sqrt{7} = 10\sqrt{7}$

d) $\sqrt{16} + \sqrt{20} = 4 + \sqrt{4 \cdot 5} = 4 + 2\sqrt{5}$

e) $\sqrt{48} - \sqrt{75} = 4\sqrt{3} - 5\sqrt{3} = -\sqrt{3}$

f) $\sqrt{20} + \sqrt{125} - \sqrt{27} = 2\sqrt{5} + 5\sqrt{5} - 3\sqrt{3} = 7\sqrt{5} - 3\sqrt{3}$

Example problems (a), (b), and (c) are as straightforward as square root simplifications can be. In them, you must attempt to find the largest perfect square that is a factor of the radicand and "pull it out of the radical." Example problems (d), (e), and (f) illustrate the important concept that radical terms can only be combined if they have identical radicand. In (d) and (f), the terms with different numbers under the radical signs <u>cannot</u> be combined as they are not like terms.

The other major radical simplification involves removing radicals from denominators. To correctly simplify these expressions, the fraction must be multiplied by a clever form of 1.[21]

Examples:
Simplify the following radical expressions.

a) $\frac{6}{\sqrt{2}}$

[20] The reason for these conventions is a bit antiquated. Decades ago, mathematicians began making numerical charts of square roots for computation purposes. With these simplification criteria, fewer roots had to be listed in these radical tables, and division by long decimals was minimized. (Remember that in the days before calculators, multiplication was preferred unanimously over division.) These reasons are no longer applicable, but square roots continue to be simplified by these rules according to convention.

[21] When trying to transform expressions in higher level mathematics, you'll find the only feasible ways to do so involve either adding clever forms of 0 (adding and subtracting the same quantity) or multiplying by clever forms of 1 (a number divided by itself).

b) $\frac{100}{\sqrt{15}}$

c) $\frac{20}{\sqrt{6}} - \frac{90}{\sqrt{54}}$

d) $\frac{10}{4+\sqrt{28}}$

Solutions:

a) To rationalize a denominator in which we find a radical of the form $\sqrt{a}$, we multiply the fraction by the clever form of 1: $\frac{\sqrt{a}}{\sqrt{a}}$. Thus, this problem can be simplified as follows.

$$\frac{6}{\sqrt{2}} = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

In the third step, we make the intuitive leap that when we square the square root of 2, we arrive at a number none other than 2 itself.

b) $$\frac{100}{\sqrt{15}} = \frac{100}{\sqrt{15}} \times \frac{\sqrt{15}}{\sqrt{15}} = \frac{100\sqrt{15}}{15} = \frac{20\sqrt{15}}{3}$$

c) $$\frac{20}{\sqrt{6}} = \frac{20}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{20\sqrt{6}}{6} = \frac{10\sqrt{6}}{3}$$

$$\frac{90}{\sqrt{54}} = \frac{90}{\sqrt{9\cdot 6}} = \frac{90}{3\sqrt{6}} = \frac{30}{\sqrt{6}} = \frac{30}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{30\sqrt{6}}{6} = 5\sqrt{6}$$

$$\frac{20}{\sqrt{6}} - \frac{90}{\sqrt{54}} = \frac{10\sqrt{6}}{3} - 5\sqrt{6} = \frac{10\sqrt{6}}{3} - \frac{15\sqrt{6}}{3} = \frac{-5\sqrt{6}}{3}$$

d) To rationalize a denominator of the form $a \pm \sqrt{b}$, we multiply the fraction by the clever form of 1: $\frac{a \mp \sqrt{b}}{a \mp \sqrt{b}}$. In this way, we multiply by the conjugate of the radical already there, and we create a difference of two squares (in a manner of speaking) so that we have $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$, and the radical disappears entirely.

$$\frac{10}{4+\sqrt{28}} = \frac{10}{4+\sqrt{28}} \times \frac{4-\sqrt{28}}{4-\sqrt{28}} = \frac{10(4-\sqrt{28})}{16-28} = \frac{40-10\sqrt{28}}{-12} = \frac{40-10\sqrt{4\cdot 7}}{-12} =$$

$$\frac{40-20\sqrt{7}}{-12} = \frac{10-5\sqrt{7}}{-3} = \frac{5\sqrt{7}-10}{3}$$

QUADRATIC EQUATIONS

Discriminant **Quadratic Equation** **Quadratic Formula**
Corollary of the Multiplication Property of Zero

Earlier, we spent a great deal of time and energy solving all types of linear equations for x. What, though, would happen if we were asked to solve the equation $x^2 = 81$? After our brief and intrepid excursion into the world of square roots in the last section, we could find an equivalent equation by taking the square root of each side, thus giving us $x = 9$. That answer has one flaw: it is WRONG. More accurately, that answer is “not entirely correct.” “Remember,” whispers the tiny voice of the benevolent math fairy, "all positive numbers have two square roots." Think for a second…yes, you've got it, the other answer here is $x = -9$ because $(-9)^2 = 81$ is also a true equation. Any equation in one variable whose highest exponent is 2 is known as a **quadratic equation**. Quadratic equations have at most two solutions. There are two major ways of solving quadratic equations. The first is solution by factoring, and the other is solution by the quadratic formula.

To solve quadratic equations by factoring, we need to think back to some of the most basic properties of multiplication. Most decathletes remember *The Multiplication Property of Zero* from beginning algebra or pre-algebra. The name may not be familiar, but most of us know that, "Zero times anything must equal zero." In more formal terms, "If $a = 0$ or $b = 0$, then $ab = 0$." There is an important principle related to this property; its names vary from textbook to textbook, but this resource will refer to it as the **Corollary of the Multiplication Property of Zero**.

Corollary of the Multiplication Property of Zero
If $ab = 0$, then either $a = 0$ or $b = 0$.

In other words, if two numbers multiply together to give 0, then at least one of those two numbers must be zero. Quite intuitively, if *three* quantities multiply together to give zero as their product, then at least one of those three numbers must be zero. It may not be apparent at first how this can apply to the solution of quadratic equations, but a few examples will drive the point home.

Examples:
Solve the following equations by factoring.
a) $x^2 – x + 12 = 0$
b) $6x^2 – x – 12 = 0$
c) $x^2 = 81$

Solutions:
a) This is a simple quadratic equation. After a bit of searching, we can factor the expression on the left to create an equivalent equation.

$x^2 – x + 12 = 0$
$(x – 4)(x + 3) = 0$ ← factor the quadratic expression on the left
$x – 4 = 0$ or $x + 3 = 0$ ← corollary of the mult. prop. of zero
$x = 4$ or $x = -3$ ← solve the appropriate equations

This gives the two possible solutions to the quadratic equation.

b) $6x^2 – x – 12 = 0$
$(2x – 3)(3x + 4) = 0$ ← factor the quadratic expression on the left
$2x – 3 = 0$ or $3x + 4 = 0$ ← corollary of the mult. prop. of zero
$x = \frac{3}{2}$ or $x = -\frac{4}{3}$ ← solve the appropriate equations

c) $x^2 = 81$
$x^2 – 81 = 0$ ← add the additive inverse of 81 to each side
$(x – 9)(x + 9) = 0$ ← factor the difference of two squares
$x – 9 = 0$ or $x + 9 = 0$ ← corollary of the mult. prop. of zero
$x = 9$ or $x = -9$ ← solve the appropriate equations

To solve quadratic equations with the quadratic formula, we will need to know and memorize the quadratic formula.[22] For a derivation of the quadratic formula and an explanation of the solving technique "completing the square," please see the first appendix.

[22] Well, duh.

The Quadratic Formula
Given a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, the solution for x is given by the formula $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$ where the radicand, $b^2 - 4ac$, is known as the **discriminant**.   If $b^2 - 4ac = 0$, the one and only solution is $x = \frac{-b}{2a}$.   If $b^2 - 4ac > 0$, there are two real solutions. (If $b^2 - 4ac$ is a perfect square, these solutions are rational.)   If $b^2 - 4ac < 0$, there are no real solutions.

Example:
Solve $6x^2 - x - 12 = 0$ using the quadratic formula.

Solution:
This is the same as example (b), which was solved above by factoring. We should get the same answers that we found above. The equation is already written in the proper form for the quadratic formula. Here, a = 6, b = -1, and c = -12.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{+1 \pm \sqrt{1 - 4(6)(-12)}}{2(6)}$$
$$x = \frac{1 \pm \sqrt{289}}{12}$$
$$x = \frac{1+17}{12} \quad \text{or} \quad x = \frac{1-17}{12}$$
$$x = \frac{18}{12} = \frac{3}{2} \quad \text{or} \quad x = \frac{-16}{12} = -\frac{4}{3}$$

Example:
Solve $x^2 - 10x + 23 = 0$ using the quadratic formula.

Solution:
This equation is already put in the proper form to apply the quadratic formula. As it is written, we have a = 1, b = -10, and c = 23.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{+10 \pm \sqrt{100 - 4(1)(23)}}{2(1)}$$
$$x = \frac{10 \pm \sqrt{8}}{2}$$
$$x = \frac{10 \pm 2\sqrt{2}}{2}$$
$$x = \frac{10 + 2\sqrt{2}}{2} \quad \text{or} \quad x = \frac{10 - 2\sqrt{2}}{2}$$
$$x = 5 + \sqrt{2} \quad \text{or} \quad x = 5 - \sqrt{2}$$

Example:
Solve the quadratic equation $x^2 = -10x - 50$ using the quadratic formula.

Solution:
The equation is not yet in the proper form to apply the quadratic formula. We transform the equation so that it becomes $x^2 + 10x + 50 = 0$. To apply the quadratic formula, we know that a = 1, b = 10, and c = 50.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-10 \pm \sqrt{100 - 4(1)(50)}}{2(1)}$$

$$x = \frac{-10 \pm \sqrt{-100}}{2}$$

This quadratic equation has no real solutions because the discriminant, the quantity under the square root sign, is negative.

APPENDIX A: COMPLETING THE SQUARE AND THE QUADRATIC FORMULA

When we want to solve an equation such as the quadratic $x^2 = 121$, what do we find for our solutions? In this case, gut instinct leads us to $x = 11$ and $x = -11$.

General Formula for a Squared Variable
If we are given that $x^2 = c$, where $c > 0$, then we solve for x by saying that $x = \pm\sqrt{c}$.

Example:
Use the general formula for a squared variable to solve $(x + 2)^2 = 50$.

Solution:
$(x + 2)^2 = 50$
$x + 2 = \pm\sqrt{50}$ ← general formula for a squared variable
$x + 2 = \pm 5\sqrt{2}$ ← simplify the square root
$x = -2 \pm 5\sqrt{2}$ ← add the additive inverse of 2 to each side
$x = -2 + 5\sqrt{2}$ or $x = -2 - 5\sqrt{2}$

We can use this idea to our advantage in an algebraic technique known as completing the square. Before we can learn how to complete the square, we have to review one concept concerning the multiplication of two binomials.

Square of a Binomial
$(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$

These formulas come from the FOIL method of multiplying binomials. If any binomial $(a \pm b)$ is multiplied by itself, then the multiplication of the binomials will result in
$(a \pm b)(a \pm b) =$
$a^2 \pm ab \pm ab + b^2 =$ ← FOIL binomial multiplication
$a^2 \pm 2ab + b^2$ ← combining like terms

If the square of a binomial can be recognized immediately and quickly, then completing the square is a feasible alternative to the quadratic formula. When we complete the square to solve a quadratic equation, we add a common constant to both sides so that the side containing the variable will become the square of a binomial. For example, if the variables and constants are separated to opposite sides of the equation, and if the "variable side" of the equation were $x^2 - 10x$, then only the lack of a constant 25 keeps that side from being the square of a binomial—adding 25 to both sides of the equation would transform the "variable side" to be $x^2 - 10x + 25$, which could afterwards be factored into $(x - 5)^2$. The general formula for a squared variable is then valid.

Example:
Solve $x^2 - 10x = 11$ by completing the square.

Solution:
Details of this problem were discussed in the paragraph above. We start by adding 25 to both sides and continue afterwards by factoring the left side into the square of a binomial.
$x^2 - 10x = 11$
$x^2 - 10x + 25 = 11 + 25$ ← add 25 to both sides to create an equivalent equation
$(x - 5)^2 = 36$ ← factor the left expression; simplify the right expression
$x - 5 = \pm 6$ ← general formula for a squared variable
$x = 5 \pm 6$ ← add the additive inverse of -5 to each side
$x = 11$ or $x = -1$

More complications arise if the coefficient of the squared term is not 1. Nevertheless, the procedure of completing the square remains completely valid. If the squared-term coefficient is not 1, first create an equivalent equation by dividing both sides of the equation by the coefficient of the squared term, then proceed as before. Remember that the constant to be added to both sides must equal the square of half the linear coefficient. Then, the variable expression can be factored into the square of a binomial and the general formula for a squared variable can be applied.

Procedure for Completing the Square
Step 1: Arrange the equation so that the variable is on one side and the constant is on the other.
Step 2: If the coefficient of the squared term is not 1, then divide both sides of the equation by that coefficient.
Step 3: Add a constant to both sides of the equation. That constant should be the square of half the linear term's coefficient.
Step 4: Factor the side containing the variable into the square of a binomial.
Step 5: Apply the general formula for a squared variable.
Step 6: Finish solving for the variable in question.

Example:
Solve $6x^2 - x - 12 = 0$ by completing the square.

Solution:

$6x^2 - x - 12 = 0$

$6x^2 - x = 12$ ← Step 1

$x^2 - \frac{1}{6}x = 2$ ← Step 2

$x^2 - \frac{1}{6}x + \frac{1}{144} = 2 + \frac{1}{144}$ ← Step 3 (half the linear term's coefficient is $\frac{1}{12}$)

$x^2 - \frac{1}{6}x + \frac{1}{144} = \frac{289}{144}$ ← express the constant side as a single fraction

$\left(x - \frac{1}{12}\right)^2 = \frac{289}{144}$ ← Step 4

$x - \frac{1}{12} = \pm\sqrt{\frac{289}{144}}$ ← Step 5

$x - \frac{1}{12} = \pm\frac{17}{12}$ ← simplify the expression on the right

$x = \frac{1}{12} \pm \frac{17}{12}$ ← Step 6

$x = \frac{18}{12} = \frac{3}{2}$ or $x = \frac{-16}{12} = -\frac{4}{3}$

Revisit the resource section on factoring and the quadratic formula to find that this equation has been solved before.

Completing the square is an alternative to the quadratic formula that at first seems hardly to be worth the effort it requires. With practice, though, the technique of completing the square can be quite fast and useful; sometimes, it is even faster than the quadratic formula.

Example:
Derive the quadratic formula. That is, solve the equation $ax^2 + bx + c = 0$ by completing the square.

Solution:

$ax^2 + bx + c = 0$

$ax^2 + bx = -c$ ← Step 1

$x^2 + \frac{b}{a}x = -\frac{c}{a}$ ← Step 2

$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$ ← Step 3 (half the linear term's coefficient is $\frac{b}{2a}$)

$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c \cdot 4a}{a \cdot 4a} + \frac{b^2}{4a^2}$ ← find a common denom. among the rational expressions

$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$ ← simplify the expression on the right

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ ← Step 4

$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ ← Step 5

$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$ ← rewrite the square root

$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$ ← simplify the square root

$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ ← Step 6

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ← simplify because of the common denominator

Thus, the solutions to the equation $ax^2 + bx + c = 0$ are $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

APPENDIX B: IMAGINARY NUMBERS

Complex Number ***i***

When mathematicians distinguish between the "real numbers" and the "imaginary numbers," many people get the idea somehow that the imaginary numbers are somehow less substantive and corporeal than the real numbers. Imaginary numbers, however, are simply numbers. For example, the concept of zero, a number representing nothing, was quite simply overlooked until the Babylonians stumbled upon it. Now, the concept of zero is second nature to most, if not all, of us.[23] In the same way, the idea of the imaginary numbers escaped mathematicians for centuries, but only a few hundred years ago, these special numbers were "discovered."

Imaginary numbers arise when the square roots of negatives are computed. If we were asked to solve the equation $x^2 = 10$, there would be two solutions: x = $\sqrt{10}$ and x = $-\sqrt{10}$. Frequently, though, mathematics arrives at equations such as $x^2 = -4$, and, until recently, mathematicians were forced to write impotently, "no solution," victims of a number system we devised for ourselves. No real number can possibly square to give a negative number. However, the number "0" arose from a natural need; when there was a lack of anything at all, *something* had to exist to signify that nothingness. In the same way, imaginary numbers arose because mathematicians realized that it was ludicrous to say that there were simply no numbers that squared to be negative. They defined one, and since the "new" number was decidedly not real, it was saddled with the unfortunate moniker "imaginary".

The imaginary unit $\boldsymbol{i}$	
$i^2 = -1$	$i = \sqrt{-1}$

Examples:
Simplify the following expressions.

a) $\sqrt{-1}$
b) $\sqrt{-16}$
c) $-\sqrt{-4}$
d) $\sqrt{-4} \times \sqrt{-12}$
e) i^7

Solutions:

a) $\sqrt{-1} = i$
b) $\sqrt{-16} = \sqrt{-1} \cdot \sqrt{16} = 4i$
c) $-\sqrt{-4} = -\sqrt{-1} \cdot \sqrt{4} = -2i$
d) $\sqrt{-4} \times \sqrt{-12} = 2i \cdot (2\sqrt{3})i = 4\sqrt{3}i^2 = -4\sqrt{3}$
e) $i^7 = (i^2)^3 i = (-1)^3 \cdot i = -i$

On the whole, you can see that the square roots of negative numbers tend to obey the same rules as the ordinary real numbers. Notice in example (d) how the negative sign was taken out of the square root before other simplification started. Make a special note that the rules for simplifying square roots do not apply when radicands are negative. Here is an incorrect way to work example (d).

Wrong Solution:

[23] … with the possible exception of parking lot attendants, who never quite seem to think that you left quickly enough.

$$\sqrt{-4} \times \sqrt{-12} = \sqrt{(-4)(-12)} = \sqrt{48} = 4\sqrt{3}$$

The negatives must be accounted for before they are multiplied away and disappear. Imaginary numbers can actually be used for those quadratic equations that we believed had no solution, those pesky quadratic equations with a negative discriminant. Now that we know how to handle it, the negative square root now simply gives us an imaginary number, and our final answer will have a dual nature: a real part and an imaginary part. A **complex number** is any number of the form $a + bi$, where a represents the number's real part, and bi represents the number's imaginary part.

Example:
Solve the quadratic equation $x^2 = -10x - 50$ over the complex numbers.

Solution:
This is the quadratic equation that we stated had no real solution earlier. It does, however, have complex solutions. Previously, we stopped when we encountered the negative discriminant. Let's press on.

$$x = \frac{-10 \pm \sqrt{-100}}{2}$$
$$x = \frac{-10 \pm 10i}{2}$$
$$x = -5 \pm 5i$$
$$x = -5 + 5i \qquad \text{or} \qquad x = -5 - 5i$$

The two solutions are not only complex numbers, they are complex conjugates.

It's important to make a special note that all real numbers and all imaginary numbers _are_ complex numbers.[24] After all, the real number "4" that we have dealt with as far back as 1st grade can technically be rewritten as "4+0i" and the imaginary number 2i that we have just learned could itself be written as "0+2i."

Arithmetic of complex numbers is rather instinctive. If we are to add two complex numbers $a + bi$ and $c + di$, then we can simply add the real parts and add the imaginary parts. If we want to subtract two complex numbers, we subtract the real parts and imaginary parts. Multiplication of complex numbers is essentially identical to multiplication of binomials; four terms result from FOIL multiplication and like terms are simplified together with the term containing i^2 made into a negative real number. These rules are restated in a table below.

The Arithmetic of Complex Numbers
Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$
Multiplication: $(a + bi) \bullet (c + di) = ac + adi + bci + bdi^2$ [FOIL] = $ac + adi + bci - bd$ = $(ac - bd) + (ad + bc)i$

Division over the complex numbers, however, is far more complicated. Much like square roots, complex numbers aren't considered "simplified" until all imaginary units are out of all denominators; also like square roots, when complex numbers are found in denominators, multiplication by a complex conjugate is the easiest way to remove the imaginary part of the denominator.

Example:

[24] … repetitive people, those mathematicians, and they're redundant too.

Substitute the solution $x = -5 + 5i$ into the quadratic equation $x^2 = -10x - 50$ and show that it satisfies the equation.

Solution:

$x^2 = -10x - 50$	
$(-5 + 5i)^2 = -10(-5 + 5i) - 50$	← substitute the given value into the equation
$25 - 50i + 25i^2 = 50 - 50i - 50$	← square the binomial, distribute the -10
$25 - 50i - 25 = 50 - 50i - 50$	← $i^2 = -1$
$-50i = -50i$	← simplify each side

The equation does indeed hold when this solution is substituted for x.

Example:

Simplify the following complex fractions, and write your answers in $a + bi$ form.

a) $\frac{1}{i}$

b) $\frac{2-i}{3+4i}$

Solution:

a) $\frac{1}{i} =$

$\frac{1}{i} \cdot \frac{-i}{-i} =$	← the complex conjugate of $0 + i$ is $0 - i$
$\frac{-i}{-i^2} =$	← perform the fraction multiplication
$\frac{-i}{-(-1)} =$	← $i^2 = -1$
$-i$	← simplify

b) $\frac{2-i}{3+4i} =$

$\frac{2-i}{3+4i} \cdot \frac{3-4i}{3-4i} =$	← the complex conjugate of $3+4i$ is $3-4i$
$\frac{6-8i-3i+4i^2}{9-16i^2} =$	← perform the binomial multiplication
$\frac{6-8i-3i-4}{9+16} =$	← $i^2 = -1$
$\frac{2-11i}{25} =$	← simplify
$\frac{2}{25} - \frac{11}{25}i$	← put answer into $a+bi$ form

Examples:
Solve the following complex equations for x.

a) $8x = 12ix + 13$
b) $3x^2 + 4ix = 12$
c) $x^2 = i$

Solutions:

a) This equation is a linear equation so we need only to solve for x.

$8x - 12ix = 13$ ← add the additive inverse of $12ix$ to each side

$x(8 - 12i) = 13$ ← factor x to try to isolate the variable

$x = \frac{13}{8-12i}$ ← multiply by the mult. inverse of $8-12i$

$x = \frac{13}{8-12i} \cdot \frac{8+12i}{8+12i}$ ← multiply by a clever form of 1 to simplify

$x = \frac{104+156i}{64-144i^2}$ ← perform the binomial multiplication

$x = \frac{104+156i}{208}$ ← $i^2 = -1$

$x = \frac{2+3i}{4}$ ← simplify the fraction

$x = \frac{1}{2} + \frac{3}{4}i$ ← put the answer in $a + bi$ form

b) This is a quadratic equation, and we will need the quadratic formula to solve.

$3x^2 + 4ix = 12$

$3x^2 + 4ix - 12 = 0$ ← put the equation into $ax^2 + bx + c = 0$ form

$x = \frac{-4i \pm \sqrt{16i^2 - 4(3)(-12)}}{2(3)}$ ← apply the quadratic formula

$x = \frac{-4i \pm \sqrt{-16-(-144)}}{6}$ ← simplify with $i^2 = -1$

$x = \frac{-4i \pm \sqrt{128}}{6}$ ← simplify

$x = \frac{-4i \pm 8\sqrt{2}}{6}$ ← simplify the radical

$x = \frac{-2i \pm 4\sqrt{2}}{3}$ ← reduce the fraction

$x = \frac{4\sqrt{2}}{3} - \frac{2}{3}i$ or $x = -\frac{4\sqrt{2}}{3} - \frac{2}{3}i$ ← put into $a + bi$ form

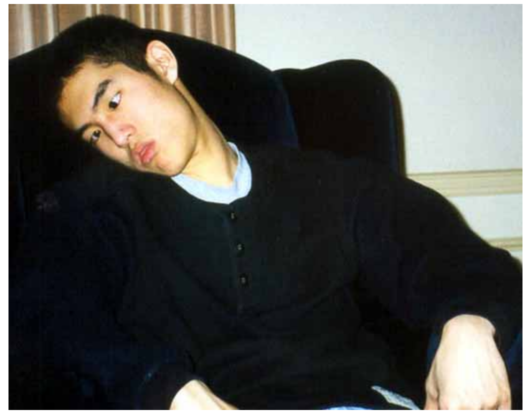
c) To solve this equation, we need to apply the general formula for a squared variable. Unfortunately, we do not have the knowledge we need to go further than $\pm\sqrt{i}$. This problem is beyond the scope of this appendix.[25]

[25] Sorry – quite a tease, huh? In case you are simply curious, then $\sqrt{i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. If you are not curious, then why are you reading this footnote?

ABOUT THE AUTHOR, PART I

Craig Chu is an alumnus of Martin High School in Arlington, Texas, and a two-year veteran of the Texas Academic Decathlon. Amidst a slew of laughs and jokes about used greeting cards and shirts right out of Where's Waldo, the team from Martin High School managed to place its highest ever in the year 2000.

Among his many strange feats, we can now say that Craig has chan…



Craig at his second-finest hour.